UNIVERSITY OF HOUSTON
MIDTERM EXAMINATION

Term: Spring      Year: 2009

Student: First Name ______________________       Last Name ______________________

UH Student ID Number ______________________

Course Abbreviation Mathematics of Signal
and Number Representations
Course Title Math 4355
Section(s) 001
Sections of Combined Course(s)
Section Numbers of Combined Course(s)
Instructor(s) B. G. Bodmann

Date of Exam March 11, 2009
Time Period Start time: 5:30pm
End time: 7:30pm
Duration of Exam 2 hours
Number of Exam Pages (including this cover sheet)
Pages 11
Exam Type Closed book
Additional Materials None
Allowed

Marking Scheme:

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1. (30 points) Let $V_0$ be the sub-space of $L^2([-\frac{1}{2}, \frac{1}{2}])$ spanned by the set of functions 
\{\text{\(g_1, g_2\)}\}, where

\[ g_1(x) = 1 \]

and

\[ g_2(x) = x. \]

(a) (10 points) Is \{\text{\(g_1, g_2\)}\} an orthonormal basis for $V_0$? Compute all the relevant quantities to explain your answer.
(b) (10 points) Let

\[ f(x) = \begin{cases} 
3, & -1/2 \leq x < 0 \\
1, & 0 \leq x \leq 1/2.
\end{cases} \]

Express the orthogonal projection \( \hat{f} \) of \( f \) onto \( V_0 \) in terms of a linear combination of \( g_1 \) and \( g_2 \).
(c) (5 points) Compute the value $\hat{f}(x)$ for $x \in [-1/2, 1/2]$.

(d) (5 points) Sketch the graphs of $f$ and of $\hat{f}$ from the preceding part on the interval $[-1/2, 1/2]$. You can put them in the same coordinate system.
2. (20 points) Let $V$ be the vector space of real-valued polynomials of degree at most two,

$$V = \{p(x) = c_0 + c_1 x + c_2 x^2, c_0, c_1, c_2 \in \mathbb{R}\},$$

and define for two functions $p, q \in V$ the “dot” product

$$\langle p, q \rangle = p(0)q(0) + p'(0)q'(0) + \frac{1}{4} p''(0)q''(0).$$

(a) (8 points) Determine all $p \in V$ for which $\langle p, p \rangle = 0$.

(b) (12 points) Is this dot product an inner product? If this is true, explain briefly why. If not, explain which of the properties of inner products is violated.
3. (20 points) Let a sequence of functions \( \{f_n\}_{n=1}^\infty \) on \([0, 1]\) be defined by

\[
f_n(x) = \begin{cases} 
\sin(nx), & 0 \leq x \leq \pi/n \\
0, & \text{else} 
\end{cases}
\]

(a) (10 points) Show that \( f_n \to 0 \) in \( L^2([0, 1]) \) as \( n \to \infty \).

(b) (10 points) Does \( f_n \to 0 \) uniformly on \([0, 1]\) as \( n \to \infty \)? Why/why not?
4. (30 points) Consider the function $f(x) = \cos(x/2)$ on $x \in [-\pi, \pi]$.

(a) (15 points) Find the Fourier coefficients $a_k$ and $b_k$ (real form of the Fourier series) for $f$. Simplify as much as possible, eliminating any trigonometric functions in your answer, if you have time.
(b) (5 points) Sketch three periods of the function to which the Fourier series converges.

(c) (10 points) Is the Fourier series uniformly convergent to \( f \) on the interval \([-\pi, \pi)\)? Quote an appropriate theorem and use it to justify your answer.
5. (20 points) Let $g(x) = x \sin(x)$ on $-\pi \leq x < \pi$.

(a) (10 points) For which points $x \in [-\pi, \pi)$ does the Fourier series converge to $g(x)$? Justify your answer by referring to conditions for pointwise convergence.

(b) (10 points) Assuming that you have (correctly) computed the partial sum of the Fourier series for $g$ as

$$S_N(x) = 1 - \frac{1}{2} \cos(x) - \sum_{k=2}^{N} \frac{2}{k^2 - 1} \cos(k\pi) \cos(kx),$$

find the value for the series

$$\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$$

by choosing an appropriate $x$ for which the Fourier series converges.
Trig formulas

\[ \sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \]

\[ \cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \]

\[ \sin^2(\alpha) = \frac{1}{2} - \frac{1}{2} \cos(2\alpha) \]

\[ \cos^2(\alpha) = \frac{1}{2} + \frac{1}{2} \cos(2\alpha) \]