UNIVERSITY OF HOUSTON
PRACTICE MIDTERM EXAMINATION

Term: Spring **Year:** 2009

Student: First Name ____________________ **Last Name** ____________________

UH Student ID Number ____

Course Abbreviation and Number	Math 4355
Course Title	Mathematics of Signal Representations
Section(s)	001
Sections of Combined Course(s)	
Section Numbers of Combined Course(s)	
Instructor(s)	B. G. Bodmann

Date of Exam	March 2009
Time Period	Start time: 5:30pm End time: 7:00pm
Duration of Exam	1 1/2 hours
Number of Exam Pages (including this cover sheet)	10 pages
Exam Type	Closed book
Additional Materials Allowed	None

Marking Scheme:

Question	**Points**	**Score**
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 points) For two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, let us define a new "dot" product with extra terms,

$$\langle\langle x, y\rangle\rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 \,.$$

(a) For any $x = (x_1, x_2)$, compute $\langle\langle x, x\rangle\rangle$. Determine for which x_1 and x_2 we obtain $\langle\langle x, x\rangle\rangle = 0$.

(b) Does $\langle\langle \cdot, \cdot\rangle\rangle$ define an inner product on $\mathbb{R}^2$? Why/why not?

2. (20 points) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on $[0,1]$ given by

$$f_n(x) = x^n \, .$$

(a) Show that $f_n \to 0$ in the norm of $L^2([0,1])$ as $n \to \infty$.

(b) Find the point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for each fixed $x \in [0,1]$.

(c) Does the sequence converge uniformly? Why/why not?

3. (25 points) Let V_N be the sub-space of $L^2([0, \pi])$ spanned by the set $\{e_1, e_2, \ldots e_N\}$, where

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx) \,.$$

   (a) Show that e_n and e_m are orthogonal for all $n \neq m$ and show that $\|e_n\|^2 = 1$ for all n.

(b) Given $f(x) = 1$, express the orthogonal projection $\hat{f}$ of f onto V_N in terms of a linear combination of e_1, e_2, $\ldots e_N$.

(c) Let $N = 4$. Compute the value $\hat{f}(x)$ for $x \in [0, \pi]$.

4. (20 points) Consider the function $f(x) = e^x + e^{-x}$ on $x \in [-\pi, \pi)$.

   (a) (15 points) Find the Fourier coefficients a_k and b_k (real form of the Fourier series) for f.

(b) (5 points) Sketch three periods of the function to which the Fourier series converges.

5. (20 points) The function $f(x) = \frac{1}{12}(\pi^2 - 3x^2)$ on the interval $[-\pi, \pi]$ has the Fourier series with partial sums

$$S_N(x) = \sum_{k=1}^{N} (-1)^{k+1} \frac{\cos(kx)}{k^2} .$$

(a) State for which $x \in [-\pi, \pi]$ the sequence $S_N(x)$ converges to $f(x)$.

(b) Use the preceding part of this problem, by choosing an appropriate x, to show

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} .$$

Trig formulas

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

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