3.6 1a) \[ W = \{ (x_1, x_2, \ldots, x_n) \mid x_1 + x_2 + \ldots + x_n = 0 \} \]

\[ W^0 = \{ f \in V^* : f(x_1, x_2, \ldots, x_n) = 0 \ \forall \ x \in W \} \]

\[ c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = 0 \]

We know \( f(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n \)

is in \( W_0 \), and \( \dim W + \dim W^0 = n \)

so by \( \dim W = n-1 \), \( \dim W^0 = 1 \)

and this \( \{ f \} \) is a basis. Thus, every \( g \in W^0 \) is \( g = cf \) for some \( c \in F \).

b) By \( \dim (W) = \dim (W^*) \)

and \( \dim (W) = n-1 \), we only have to find a basis for \( W^* \) consisting of functionals \( \{ f_j \}_{j=1}^{n-1} \) such that for each \( j \),

\[ f_j(x_1, \ldots, x_n) = c_1^{(j)} x_1 + \ldots + c_n^{(j)} x_n \]

and \( c_1^{(j)} + c_2^{(j)} + \ldots + c_n^{(j)} = 0 \). Moreover, \( n-1 \) linear indep. functionals are enough.
Given a basis \( \{f_j^i\}^{n-1}_{j=1} \) for \( \mathcal{W}^* \),

\[
f_j^i(x_1, \ldots, x_n) = c_1^i x_1 + \ldots + c_n^i x_n
\]

Let

\[
d_j = c_1^j + c_2^j + \ldots + c_n^j.
\]

Now consider

\[
\tilde{f}_j^i(x_1, \ldots, x_n) = \left( c_1^i \frac{d_j - d_j^-}{n} \right) x_1 + \ldots + \left( c_n^i \frac{d_j - d_j^-}{n} \right) x_n
\]

then

\[
\tilde{c}_1^j + \tilde{c}_2^j + \ldots + \tilde{c}_n^j = d_j - d_j^- = 0.
\]

Moreover, for \( x \in \mathcal{W} \), \( f_j^i(x) = \tilde{f}_j^i(x) \).

Since \( \dim \mathcal{W}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) : \tilde{c}_1 + \tilde{c}_2 + \ldots + \tilde{c}_n = 0 \)

\( = n-1 \)

the \( \tilde{c}_j \) are unique.
3.7 \ a) \ g(x_1, x_2) = (f \circ T)(x_1, x_2) \\
= f(T(x_1, x_2)) \\
= f(x_1, 0) = ax_1 \\

b) \ g(x_1, x_2) = f(T(x_1, x_2)) \\
= f(-x_2, x_1) \\
= -ax_2 + bx_1 \\
c) \ g(x_1, x_2) = f(T(x_1, x_2)) \\
= f(x_1 - x_2, x_1 + x_2) \\
= a(x_1 - x_2) + b(x_1 + x_2) \\
= (a + b)x_1 + (b - a)x_2 \\

6 \ \text{Let } D^+: P_n(\mathbb{R})^* \to P_n(\mathbb{R})^* \text{ by } D^+(g) = g \circ D \text{ for all } g \in P_n(\mathbb{R})^*. \text{ So } \ker(D^+) = \{ g \in P_n(\mathbb{R})^* : g(p') = 0 \ \forall \ p \in P_n(\mathbb{R}) \} \text{ for all } p' \text{'s, or } g \in (\ker(D))^0. \text{ But } \text{ran}(D) = P_{n-1}(\mathbb{R}), \text{ so } \\
g(c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}) = 0 \\
\text{as proved in class. Now, } \dim (\ker(D))^0 + \dim (\text{ran}(D)) = n \Rightarrow \dim (\text{ran}(D))^0 = 1.
We only need to find one functional.

Given the standard basis \( f_1, x, x^2, \ldots, x^n \)
and its dual \( f_1^*, f_2^*, \ldots, f_n^* \), this is, e.g.
\( f_n \), b/c \( f_n(x^j) = 0 \quad \forall j \leq n-1 \). So \( \{f_j^*\} \) is a basis for \( \ker(D) \).

5.21 (a) Let \( A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_2 = A_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c \neq 1 \)
then \( D(cA_1, A_2, A_3) = c + 1 \)
\[ + cD(A_1, A_2, A_3) = c(c+1) = 2c \]
Not linear in \( A_1 \) = not 3-linear.

(b) Same \( A_1, A_2, A_3 \) as in (a), \( c^2 - c \neq 0 \)
\[ D(cA_1, A_2, A_3) = c^2 + 3c \]
\[ + cD(A_1, A_2, A_3) = c(1+3) \]
Not 3-linear.

(c) \( D(A_1, A_2, A_3) = A_{12} A_{23} \)
Linearity in \( A_1 \):
\[ D(0, A_2, A_3) = 0 \]
\[ D(cA_1 + B_1, A_2, A_3) = (cA_{11} + B_{11}) A_{22} A_{33} \]
Linear in \( A_2 \):
\[ D(A_1, A_2 + B_2, A_3) = cD(A_1, A_2, A_3) + D(B_2, A_2, A_3) \]
Similar for \( A_2, A_3 \) = 3-linear.
d) Pick \( A_1 = A_2 = A_3 = \left( \begin{array}{c}
1 \\
1 \\
1
\end{array} \right) \), then

\[ D(A_1, cA_2, A_3) = (1)(c)(c) + 5(c)(c)(c) \]

\[ = c^2 + 5c^3 \]

\[ \frac{d}{dc} D(A_1, A_2, A_3) = 6c \quad \text{for e.g.} \quad c = \frac{1}{2} \]

\[ \frac{1}{4} + \frac{5}{8} = \frac{7}{8} < \frac{6}{2} = 3 \]