1. Let $A$ be an $m \times m$ matrix over a field $F$ and suppose there is an $m \times m$ matrix $B$ such that $BA = I_m$. Prove that $A$ is invertible and that $B = A^{-1}$.

2. Let $V = \mathbb{R}^{1 \times 2}$, the set of all $1 \times 2$ matrices with real entries. Define:
   vector addition: $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 & 0 \\ x_2 + y_2 & 0 \end{bmatrix}$
   scalar multiplication: $c \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} cx_1 & cx_2 \end{bmatrix}$ where $c \in \mathbb{R}$.
Which of the axioms of a vector space are satisfied and which are not. Justify your answers.

3. Let $V$ be a vector space over the field $F$, $\alpha \in V$, $c \in F$.
Show that if $c \alpha = \theta$ then either $c = 0$ or $\alpha = \theta$.

4. Let $V$ be vector space over the reals consisting of all continuous real-valued functions defined on the closed interval $[0,1]$, with the usual operations of addition and scalar multiplication of functions. Let $W$ consist of all functions $f(t)$ in $V$ for which $\int_0^1 f(t) \, dt = 0$.
Show that $W$ is a subspace of $V$. 

Math 4377

Homework due Sep 16, 2008

1. Let $A, B$ be $m \times n$ matrices, and assume $BA = I_m$, then for any $m \times n$ matrix $X$, $AX = 0$ implies $X = 0$

b/c $AX = 0 \Rightarrow BAX = 0 \Rightarrow X = 0$.

Now, by Thm 13, we know $A$ is invertible. Moreover, the right inverse, $C$, is identical to $B$, because

$BA = I_m = AC$

$\Rightarrow B = BI_m = BAC$

$= I_m C = C$.

2. Axioms:

   • addition $[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, 0]$

   $\Rightarrow [y_1 + x_1, 0] = [y_1, y_2] + [x_1, x_2]$

   commutative!

   • addition $[x_1, x_2] + ([y_1, y_2] + [z_1, z_2])$

   $= [x_1, x_2] + [y_1 + z_1, 0]$

   $= [x_1 + y_1 + z_1, 0]$

   $= [x_1 + y_1, 0] + [z_1, z_2]$

   $= ([x_1, x_2] + [y_1, y_2]) + [z_1, z_2]$
- Unique zero vector

\[
\begin{bmatrix} x_1, x_2 \end{bmatrix} + \begin{bmatrix} y_1, y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1, 0 \end{bmatrix} \\
\neq \begin{bmatrix} x_1, x_2 \end{bmatrix}
\]

unless \( x_2 = 0 \), fails!

- Unique negative

\[
\begin{bmatrix} x_1, x_2 \end{bmatrix} + \begin{bmatrix} y_1, y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1, 0 \end{bmatrix} = \begin{bmatrix} 0, 0 \end{bmatrix}
\]

\( y_1 = -x \)

\( y_2 \) can be anything

uniqueness fails!

- Scalar multiplication

\[
c \begin{bmatrix} x_1, x_2 \end{bmatrix} = \begin{bmatrix} cx_1, cx_2 \end{bmatrix}
\]

has desired properties because it is as in lecture.
3. Let \( \alpha \in V, \ c \in F \) and \( cx = 0 \).

Assume both \( c \) and \( \alpha \) are non-zero, then \( \frac{1}{c}(cx) = \alpha \), but also

\[
\frac{1}{c}(c\alpha) = \frac{1}{c}(0) = 0
\]

which is a contradiction. Consequently, either \( c = 0 \) or \( \alpha = 0 \).

4. We check two properties:

- \( f(t) = 0 \) is in \( V \), \( b(c) \)

\[
\int_0^1 f(t) \, dt = \int_0^1 0 \, dt = 0
\]

- If \( f, g \in V \) and \( c \in F \), then \( \alpha \) is \( f + cg \), because

\[
\int_0^1 (f(t) + cg(t)) \, dt
\]

\[
= \int_0^1 f(t) \, dt + c \int_0^1 g(t) \, dt
\]

\[
= 0 + c(0) = 0
\]
Homework Set 5, due Thursday, Sep 25, 1pm

Section 2.2

1 Which of the following sets of vectors $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ ($n \geq 3$)? If yes, show it has the properties required of a subspace; if not, find an example that shows how a property of subspaces is violated:

(a) all $\alpha$ such that $a_1 \geq 0$;
(b) all $\alpha$ such that $a_1 + 3a_2 = a_3$;
(c) all $\alpha$ such that $a_2 = a_1^2$;
(d) all $\alpha$ such that $a_1a_2 = 0$;
(e) all $\alpha$ such that $a_2$ is rational.

4 Let $W$ be the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfy

\[
2x_1 - x_2 + \frac{1}{3}x_3 - x_4 = 0 \\
x_1 + \frac{2}{3}x_3 - x_5 = 0 \\
9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0.
\]

Find a finite set of vectors which spans $W$.

5 Let $F$ be a field and let $n$ be an integer with $n \geq 2$. Let $V$ be the vector space of all $n \times n$ matrices over $F$. Which of the following sets of matrices $A$ are subspaces of $V$? If yes, show it has the properties required of a subspace; if not, show how a property of subspaces is violated:

(a) all invertible $A$;
(b) all non-invertible $A$;
(c) all $A$ such that $AB = BA$ where $B$ is a fixed matrix in $V$;
(d) all $A$ such that $A^2 = A$.

Section 2.3

2 Are the vectors $\alpha_1 = (1, 1, 2, 4), \alpha_2 = (2, -1, -5, 2), \alpha_3 = (1, -1, -4, 0), \alpha_4 = (2, 1, 1, 6)$ linearly independent in $\mathbb{R}^4$?

3 Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ in the preceding exercise.

4 Show that the vectors $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1)$, and $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors $\epsilon_1, \epsilon_2$ and $\epsilon_3$ as a linear combination of $\alpha_1, \alpha_2$, and $\alpha_3$. 
Math 4377

Homework Set 5,
due Sept. 25, 2008

2.2 1. a) \( S = \{ (a_1, \ldots, a_n) : a_i > 0 \} \)

is not a subspace, b/c

\((a_i, 0, \ldots, 0) \in S \) for \( a_i > 0 \)

but not

\(- (a_i, 0, \ldots, 0) = (-a_i, 0, \ldots, 0) \).

b) \( S = \{ (a_1, \ldots, a_n) : a_1 + 3a_2 = a_3 \} \)

i) \((0,0,\ldots,0) \in S \checkmark \)

ii) if \( \alpha, \beta \in S \) then for \( c \in \mathbb{F} \)

\( a_1 + 3a_2 = a_3 \)

\( cb_1 + 3cb_2 = cb_3 \)

so

\( (a_1 + cb_1) + 3(a_2 + cb_2) = a_3 + cb_3 \)

which means

\( (a_1 + cb_1, a_2 + cb_2, \ldots) \in S \)

\( \alpha + c\beta \in S \).

So \( S \) is a subspace.
c) \[ S = \{(a_1, \ldots, a_n) : a_2 = a_1^2\} \]

We have \((1, 1, 0, \ldots, 0) \in S\)
but not \((c, c, 0, \ldots, 0)\)
for \(c \in \{0, 1\}\), bc then \(c = c^2\),
so \(S\) is not a subspace.

d) \[ S = \{(a_1, a_2, \ldots, a_n) : a_1a_2 = 0\} \]

We have \((1, 0, 0, \ldots, 0) \in S\)
and \((0, 1, 0, \ldots, 0) \in S\)
but \((1, 1, 0, \ldots, 0) \notin S\)
\[ = (1, 0, \ldots, 0) + (0, 1, \ldots, 0), \]
so \(S\) is not a subspace.

e) \[ S = \{(a_1, a_2, \ldots, a_n) : a_2 \in \mathbb{Q}\} \]

We have \((0, 1, 0, \ldots, 0) \in S\)
but \((0, \pi, 0, \ldots, 0) \notin S\)
\[ = \pi (0, 1, 0, \ldots, 0) \]
so \(S\) is not a subspace.
4. Let $(x_1, x_2, \ldots, x_5) \in \mathbb{R}^5$ satisfy

\[
\begin{align*}
2x_1 - x_2 + \frac{4}{3}x_3 - x_4 &= 0 \\
x_1 + \frac{2}{3}x_3 - x_5 &= 0 \\
9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0
\end{align*}
\]

Row reduce

\[
\begin{pmatrix}
2 & -1 & \frac{4}{3} & 0 & -1 \\
1 & 0 & \frac{2}{3} & 0 & -1 \\
9 & -3 & 6 & -3 & -3
\end{pmatrix}
\]

\[
\sim
\begin{pmatrix}
1 & 0 & \frac{2}{3} & 0 & -1 \\
0 & 1 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

so after choosing $x_3, x_4, x_5$, we have determined values of $x_1, x_2$

\[
\begin{align*}
x_1 &= -\frac{2}{3}x_3 + x_5 \\
x_2 &= -x_4 + 2x_5.
\end{align*}
\]

Spanning set is

\[
\{(\frac{-2}{3}, 0, 1, 0, 0), (0, -1, 0, 1, 0), (1, 2, 0, 0, 1)\}.
\]
5. a) $S = \{ A \in \mathbb{F}^{n \times n} : A \text{ invertible} \}$
   is not a vector space b/c
   0 (zero matrix) is not included
b) $S = \{ A \in \mathbb{F}^{n \times n} : A \text{ non-invertible} \}$

Let

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$D_1, D_2$ are not invertible

but

$$D_1 + D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is invertible

Not closed under addition.
c) \( S = \{ A \in F^{n \times n} : A \mathbf{B} = \mathbf{B} A \} \) for fixed \( \mathbf{B} \)

i) \( 0 \) matrix is in \( S \)

ii) given \( A_1, A_2 \in S \)

we know \( A_1 \mathbf{B} = \mathbf{B} A_1 \)

\( A_2 \mathbf{B} = \mathbf{B} A_2 \),

so

\[
(A_1 + c A_2) \mathbf{B} = A_1 \mathbf{B} + c A_2 \mathbf{B} = B A_1 + c B A_2 = B (A_1 + c A_2)
\]

which means \( A_1 + c A_2 \in S \)

\( \Rightarrow S \) is a subspace!

d) \( S = \{ A \in F^{n \times n} : A^2 = A \} \)

\[
D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S
\]

but

\[
2D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}
\]

gives \( (2D)^2 = 4D = 2(2D) \)

so \( 2D \notin S \) \( \Rightarrow \) not a subspace.
2.3 2. $\alpha_1 = (1, 1, 2, 4)$, $\alpha_2 = (2, -1, -5, 2)$
$\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$

\[
\begin{pmatrix}
1 & 1 & 2 & 4 \\
2 & -1 & -5 & 2 \\
1 & -1 & -4 & 0 \\
2 & 1 & 1 & 6
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 2 & 4 \\
2 & -1 & -5 & 2 \\
0 & -2 & -6 & -4 \\
0 & 2 & 6 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 2 & 4 \\
2 & -1 & -5 & 2 \\
0 & -2 & -6 & -4 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

does not have full rank, so vectors are linearly dependent.

3. Continue eliminating rows

\[
\begin{pmatrix}
1 & 1 & 2 & 4 \\
0 & -3 & -9 & -6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

so basis vectors are

$B = \{ (1, 1, 2, 4), (0, 3, 9, 6) \}$
4. Vectors $\mathbf{x}_1 = (1, 0, -1), \mathbf{x}_2 = (1, 2, 1), \mathbf{x}_3 = (0, -3, 2)$

$$
\begin{pmatrix}
1 & 0 & -1 \\
1 & 2 & 1 \\
0 & -3 & 2
\end{pmatrix} \sim
\begin{pmatrix}
1 & 0 & -1 \\
0 & 2 & 2 \\
0 & -3 & 2
\end{pmatrix} \sim
\begin{pmatrix}
1 & 0 & -1 \\
0 & 2 & 2 \\
0 & 0 & 5
\end{pmatrix}
$$

has maximal rank $\Rightarrow \exists \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \not\in \mathbb{R}^n$ indp.

Want to find $(P_{ij})_{1 \leq i, j \leq 3}$ so that $E_j = \sum_{i=1}^{3} P_{ij} \mathbf{x}_i$.

We get:

$$
P = \begin{pmatrix}
\frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\
\frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\
\frac{1}{5} & -\frac{1}{5} & \frac{1}{5}
\end{pmatrix}
$$

so

$$
E_1 = \frac{7}{10} \mathbf{x}_1 + \frac{3}{10} \mathbf{x}_2 + \frac{1}{5} \mathbf{x}_3
$$

$$
E_2 = -\frac{1}{5} \mathbf{x}_1 + \frac{1}{5} \mathbf{x}_2 - \frac{1}{5} \mathbf{x}_3
$$

$$
E_3 = -\frac{3}{10} \mathbf{x}_1 + \frac{3}{10} \mathbf{x}_2 + \frac{1}{5} \mathbf{x}_3
$$