Math 6320 - Practice Midterm Exam - Fall 2010

1. Let \((X, M, \mu)\) be a measure space, let \(\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}\) be two sequences of non-negative measurable functions and assume \(f_n(x) \leq g_n(x)\) for all \(n \in \mathbb{N}\), as well as point-wise convergence \(f_n(x) \to f(x)\) and \(g_n(x) \to g(x)\), for all \(x \in X\). Assume that all \(\int g_n d\mu\) and \(\int g d\mu\) are finite, and that \(\int g_n d\mu \to \int g d\mu\).

(a) Quote a famous result from class to establish that \(\liminf_n \int f_n d\mu \geq \int f d\mu\).

(b) Prove \(\limsup_n \int f_n d\mu \leq \int f(x) d\mu\). Hint: consider \(h_n = g_n - f_n\).

(c) What can you say about \(\lim_n \int f_n d\mu\)?

2. Let \((X, M, \mu)\) be a measure space and let \(0 < c < \infty\).

(a) Let \(f \geq 0\) be a measurable function on \(X\) such that \(\int_E f d\mu \leq c \mu(E)\) for all \(E \in M\). If \(\mu(X) < \infty\), prove that \(\mu(\{x \in X : f(x) > c\}) = 0\).

(b) Let \(g \geq 0\) be a measurable function on \(X\) such that \(\mu(\{x \in X : g(x) > c\}) = 0\). Prove that then \(\int_E g d\mu \leq c \mu(E)\) for all \(E \in M\).

3. Let \(\mu\) be a regular Borel measure on a compact Hausdorff space \(X\) and assume \(\mu(X) = 1\). Prove that there is a compact set \(K \subset X\) such that \(\mu(K) = 1\) and \(\mu(H) < 1\) for every compact, proper subset \(H\) of \(K\) (this means for each compact \(H \subset K\) with \(H^c \cap K \neq \emptyset\)).

4. State Lusin’s theorem (without proving it).