Assignment 2, due Friday, September 7, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let \( p \in \mathbb{N}, b > 0 \) and assume \( u \) is the solution of the integral equation

\[
    u(x) = \int_0^x \sin(u(t))(u(t))^p \, dt
\]
on the interval \([-b, b]\).

1. Let \( M = \sup_{-b \leq x \leq b} |u(x)| \). Prove that for each integer \( n \geq 0 \), \( |u(x)| \leq M^np|x|^n/n! \). Hint: \( |\sin(y)| \leq |y| \).

2. Use the preceding part to show that \( u = 0 \).

Problem 2

Consider the initial value problem

\[
    y'(x) = x^2 + (y(x))^2, y(0) = 0
\]

1. Show that this differential equation satisfies a local Lipschitz condition (in the second variable) on the set \( Q = [0, b] \times [-R, R] \), but not on the set \([0, b] \times \mathbb{R} \).

2. Integrate the inequality \( y'(x) \geq 1 + (y(x))^2 \) to prove that the solution to the initial value problem grows above any bound in finite time.

Problem 3

Let \( y \) be the solution to the initial value problem \( y'(x) = e^{xy(x)} \) and \( y(0) = 1 \) for \( x \in [-1/2, 1/2] \). Suppose you wish to compare this with the solution \( y_n \) to the initial value problem \( y'(x) = \sum_{k=0}^n \frac{(xy(x))^k}{k!}, y_n(0) = 1 \), on \([-1/2, 1/2] \).

1. Show that as \( n \to \infty \), \( y_n \to y \) uniformly on \([-1/2, 1/2] \).

2. Find \( n \) so that \( d(y, y_n) < 0.0001 \).
Problem 4

Let $y$ be a solution of the initial value problem $y'(x) = h(x, y(x))$ and $y(a) = y_0$, where $h$ is continuous on $[a, b] \times \mathbb{R}$ and $L$-Lipschitz in the second variable. Assume $\eta$ is a differentiable function satisfying $|\eta'(x) - h(x, \eta(x))| \leq \epsilon$ for each $x \in [a, b]$ and $|\eta(a) - y_0| \leq \delta$. Show that for $x \in [a, b]$,

$$|y(x) - \eta(x)| \leq \delta e^{L(x-a)} + \frac{\epsilon}{L} (e^{L(x-a)} - 1).$$

Hint: Find a variation of the proof for stability of solutions.