Assignment 4, due Friday, September 21, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Use the implicit function theorem to show that the system of equations

\[
\begin{align*}
3x + y - z + u^2 &= 11 \\
x - y + 2z + u &= -2 \\
2x + 2y - 3z + 2u &= 15
\end{align*}
\]

can be solved locally for \((y,z,u)\) as function of \(x\). Give the formulas for the derivatives \(dy/dx\), \(dz/dx\), and \(du/dx\). The point \((x_0,y_0,z_0,u_0) = (1,3,-1,2)\) solves the equations. Use the derivatives to give a linear approximation to the nearby point \((0.9,y,z,u)\) that satisfies the equations.

Problem 2

Let \(f: \mathbb{R}^3 \rightarrow \mathbb{R}\) be continuously differentiable. Assume at the point \((x_0,y_0,z_0)\) the partial derivatives satisfy \(\partial f(x_0,y_0,z_0)/\partial x \neq 0\), \(\partial f(x_0,y_0,z_0)/\partial y \neq 0\) and \(\partial f(x_0,y_0,z_0)/\partial z \neq 0\). Explain that there are functions \(g_i\) with \(x = g_1(y,z)\), \(y = g_2(x,z)\) and \(z = g_3(x,y)\) that are defined in some open sets in \(\mathbb{R}^2\) and \(f(g_1(y,z),y,z) = f(x,g_2(x,z),z) = f(x,y,g_3(x,y)) = f(x_0,y_0,z_0)\) and if \((x,y,z)\) has coordinates in these open sets, then

\[
\frac{\partial g_1}{\partial y} \frac{\partial g_2}{\partial z} \frac{\partial g_3}{\partial x} = -1.
\]

Verify this identity for the partial derivatives for the example of a van der Waals gas with \(f(p,V,T) = pV - RT = f(p_0,V_0,T_0)\), where \(p_0, V_0, V_0 > 0\) and \(R > 0\) is a constant.

Problem 3

Let \(y\) be implicitly given as a function of \(x\) by \(f(x,y) = e^y + y^3 + x^3 + x^2 = 1\). Explain why this equation defines \(y = g(x)\) globally on \(\mathbb{R}\). Find the values of \(x\) for which \(g'(x) = 0\). Verify that if \(g'(x) = 0\), then

\[
g''(x) = -\frac{1}{\partial f(x,y)/\partial y} \frac{\partial^2 f(x,y)}{\partial x^2}.
\]

Use this to decide for which \(x\) the function \(g\) has a local maximum/minimum.
Problem 4

Let $A$ be a symmetric, real $n \times n$ matrix, so $A_{i,j} = A_{j,i}$ for $i, j \in \{1, 2, \ldots, n\}$. Consider the corresponding quadratic function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \sum_{i,j=1}^{n} A_{i,j}x_i x_j$. Show that on the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ in $\mathbb{R}^n$, $f$ assumes its maximum at a point $x$ for which $Ax = \lambda x$ with some $\lambda \in \mathbb{R}$. 