MATH 6360
Applied Analysis
Fall 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 5, due Friday, October 5, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let (X, d) and (Y, ρ) be metric spaces with completions (C, d) and (D, ρ), assuming $X \subset C$ and $Y \subset D$. Prove that the metric the space $(X \times Y, \sigma)$ with the metric $\sigma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$ has the completion $(C \times D, \sigma)$.

Problem 2

Show that the Hölder inequality for $f, g \in C([a, b])$ and $1 < p < \infty$ cannot be improved because for any $f \in C([a, b])$, there is $g \in C([a, b])$ such that $\int_a^b fg dx = \|f\|_p \|g\|_q$, $1/p + 1/q = 1$. Hint: For $x \in [a, b]$ with $f(x) \neq 0$, set $g(x) = |f(x)|^p / f(x)$ and if $f(x) = 0$ set $g(x) = 0$.

Problem 3

Let U be an open set in the interval $[a, b]$. Show that the distance of any point x in U from the complement $U^c = [a, b] \setminus U$, given by $d(x, U^c) = \inf_{y \in U^c} d(x, y)$, is a continuous function on U. (Hint: U^c is closed.)

Show that the characteristic function χ_U of an open set $U \subset [a, b]$ is the limit of an increasing sequence of continuous functions. Here, $\chi_U(x) = 1$ if and only if $x \in U$ and otherwise $\chi_U(x) = 0$. Hint: Use the distance function to construct such a sequence.

Consequently, deduce that the characteristic function of any open set $U \subset [a, b]$ is in $L^1([a, b])$.

Problem 4

Define a map $T : C([0, 1]) \to C([0, 1])$ by

$$Tf(x) = \int_0^1 k(x, y) f(y) dy$$

where $k : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Show that the operator norm $\|T\|$ equals

$$\|T\| \equiv \sup\{\|Tf\|_\infty : f \in C([0, 1]), \|f\|_\infty \leq 1\} = \max_{0 \leq x \leq 1} \int_0^1 |k(x, y)| dy \, .$$