Assignment 6, due Friday, October 19, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $C([-1, 1])$ be equipped with the norm $\|f\|_1 = \int_{-1}^{1} |f(x)|\,dx$. Does the linear functional $F : f \mapsto f(0)$ extend uniquely to $L^1([-1, 1])$?

Problem 2

Show that if $X$ and $Y$ are normed spaces, and $B(X, Y)$ is a Banach space, then $Y$ is a Banach space. Hint: If $F \in X^*$ is not the zero functional, consider $T : Y \to B(X, Y)$ given by $T(y)x = F(x)y$.

Problem 3

Let $c_0$ be the normed vector space containing each sequence $x = (x_n)_{n \in \mathbb{N}}$ with $\lim_n x_n = 0$, equipped with the norm $\|x\|_\infty = \sup_n |x_n|$ for $x \in c_0$. Show that the dual space $c_0^*$ is isometrically isomorphic to $\ell^1$, the space of summable sequences, equipped with $\|y\|_1 = \sum_{n=1}^{\infty} |y_n|$ for $y \in \ell^1$.

Problem 4

Let $\text{Pol}([0, 1])$ be the space of real polynomials on $[0, 1]$, equipped with $\|p\|_1 = \int_0^1 |p(x)|\,dx$. Define for $p \in \text{Pol}([0, 1])$ the map $T(p)q = \int_0^1 p(x)q(x)\,dx$, then show $T(p)$ is continuous, so $T : \text{Pol}([0, 1]) \to \text{Pol}([0, 1])^*$. Moreover, show that if $A = \{p \in \text{Pol}([0, 1]) : \|p\|_1 \leq 1\}$, then for each $q \in \text{Pol}([0, 1])$, $\sup_{p \in A} |T(p)q|$ is finite, but $T$ is not uniformly bounded. Why does this not contradict the Banach-Steinhaus theorem?