Final Practice Exam – Math 6360  
December, 2018

First name: ______________ Last name: ________________ Last 4 digits ID: __________

1 Memorization

a. State the contraction mapping theorem on metric spaces.

b. State a theorem on the existence and uniqueness of initial-value problems of first-order differential equations on an interval \([a, b]\).
c. State carefully the definition of the open mapping theorem for linear transformations $T : X \rightarrow Y$, with appropriate space $X$ and $Y$.

d. State under which condition the norm on a normed real vector space is induced by an inner product.

Turn in this first part to obtain the next portion of the exam.
In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Let $X = C([0, a])$, equipped with the metric induced by the norm $\|f\|_{\infty} = \sup_{0 \leq x \leq a} |f(x)|$.

a. Show that the averaging operator $A$ defined by $Af(0) = f(0)$ and

$$Af(x) = \frac{1}{x} \int_0^x f(x) \, dx, \quad x > 0$$

is a bounded linear operator mapping $X$ to itself with $\|A\| \leq 1$. 
b. Does $A$ have any fixed points? If so, find one. If not, explain why not.
3 Problem

Let \( B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\} \). Find the maximum value of \( \| (x, y, z) \|_p = (|x|^p + |y|^p + |z|^p)^{1/p} \) for \( 1 < p < 2 \) occurring among all points in \( B \). Distinguish cases if necessary.
4 Problem

Let the set $A \subset \ell^\infty$ be given by

$$A = \{ x = (x_1, x_2, x_3, \ldots) \in \ell^\infty : 0 < x_n < 1, n \in \mathbb{N} \}.$$

Decide whether this set is open in $\ell^\infty$, equipped with the metric induced by the sup-norm. Explain the reasons for your answer.
5 Problem

Is the sequence of functions \((f_n)_{n=1}^\infty\) in \(C([0, 1])\), equipped with the sup-norm, given by

\[ f_n(t) = \sqrt{n}(t^n - t^{n+1}) \]

convergent in \(C([0, 1])\)?
6 Problem

Let $X$ be a uniformly convex normed linear space, $F \in X^*$, $F \neq 0$. Consider the set $Y = \{y \in X : F(y) = 1\}$. Prove that

$$\|F\| = \frac{1}{\inf_{y \in Y} \|y\|}.$$
7 Problem

Let $H$ be the Hilbert space $l^2$ and $S$ the map defined by

$$(Sx)_k = x_k / k, \ k \in \mathbb{N}$$

for any $x = (x_1, x_2, \ldots) \in l^2$. Show that $S$ is a bounded map from $l^2$ to $l^2$, and that it is one-to-one. Is the range of $S$ a closed subspace of $l^2$? Explain your answer.
8 Problem (20 points)

Let $X$ be the space $C^1([0,1])$ containing all continuously differentiable functions. Show that the two norms

\[ \|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty \]

and

\[ \|f\|_0 = |f(0)| + \|f'\|_\infty \]

are equivalent.