Assignment 1, due Thursday, September 2, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

**Problem 1**

Let $\mathbb{Z}_2 = \{0, 1\}$ and for $n \in \mathbb{N}$, the set $X = \mathbb{Z}_2^n$ contain all $x = (x_1, x_2, \ldots, x_n)$ with entries $x_j \in \mathbb{Z}_2$, $1 \leq j \leq n$. Define a metric $d$ on $X$ by $d(x, y) = \sum_{j=1}^{n} |x_j - y_j|$.

a. Show that for any $x \in X$, the closed ball $\overline{B}_r(x)$ of radius $r \in \mathbb{N}$, centered at $x$, has at least $(n/r)^r$ elements. (Hint: First explain why the size of the ball is larger than $(n^r_r) = n(n-1) \cdots (n-r+1) r^{r-1} \cdots 1$.)

b. Show that for $r \in \mathbb{N}$, if $x$ and $x'$ have distance $d(x, x') > 2r$, then $\overline{B}_r(x)$ and $\overline{B}_r(x')$ are disjoint.

c. Let $r \in \mathbb{N}$ and $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ have mutual distance $d(x^{(i)}, x^{(i')}) \geq 2r + 1$ for all $i \neq i'$. By counting elements, prove that $k \leq \left(\frac{r}{n}\right)^r 2^n$.

**Problem 2**

Explain why the function $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$

does not have a fixed point. Prove that for all $x, y \in \mathbb{R}$,

$$|T(x) - T(y)| < |x - y|.$$  

Why does this example not contradict the contraction mapping theorem?

**Problem 3**

For $b > 0$ and $a \in \mathbb{R}$, define $T$ on $C([0, b])$ by $Tf(x) = a + \int_{0}^{x} f(t)xe^{-xt}dt$. Prove that $T$ is a contraction when $C([0, b])$ is equipped with the metric $d_\infty(f, g) = \max_{0 \leq t \leq b} |f(t) - g(t)|$. Use this fact to deduce that there is a unique solution $f \in C([0, \infty))$ to the integral equation $f(x) = a + \int_{0}^{x} f(t)xe^{-xt}dt$, $x \geq 0$.

**Problem 4**

An $n \times n$ real matrix $A$ is said to be diagonally dominant if for each row the sum of the absolute value of off-diagonal terms in this row is strictly less than the value of the diagonal term in this row. Write $A = D - L - U$, where $D$ is a diagonal matrix, $L$ lower triangular and $U$ upper triangular (both $L$ and $U$ with zero diagonal). Let the row-sum norm $\|X\|_\infty$ of a matrix $X = (X_{i,j})_{i,j=1}^{n}$ be defined by

$$\|X\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |X_{i,j}|.$$
then show that if $A$ is diagonally dominant, $\|L + U\|_\infty < \|D\|_\infty$. Consider $\mathbb{R}^n$ as a metric space with the (non-Euclidan) metric $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$. If $A$ is diagonally dominant, prove that for any $b \in \mathbb{R}^n$, a solution to the linear equation $Ax = b$ can be obtained as the limit of the sequence $(x^{(n)})_{n=1}^{\infty}$ defined by

$$x^{(n+1)} = D^{-1}(L + U)x^{(n)} + D^{-1}b$$

and this solution is unique. Thus, explain why any diagonally dominant $n \times n$ matrix $A$ is invertible.