Assignment 5, due Thursday, October 14, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

**Problem 1**

Let $(X,d)$ and $(Y,\rho)$ be metric spaces with completions $(C,d)$ and $(D,\rho)$, assuming $X \subset C$ and $Y \subset D$. Prove that the metric the space $(X \times Y,\sigma)$ with the metric $\sigma((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2),\rho(y_1,y_2)\}$ has the completion $(C \times D,\sigma)$.

**Problem 2**

Let $c_{0,0}$ be the space of sequences that are eventually zero, so for each $x = (x_1, x_2, \ldots) \in c_{0,0}$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = 0$. Equip this space with the metric $d_\infty$. Show that the completion of $c_{0,0}$ is the space $c_0$, containing each sequence $x$ with $\lim_{n \to \infty} x_n = 0$. Hint: You know $c_{0,0} \subset \ell_\infty$ and that $\ell_\infty$ is complete.

**Problem 3**

Let $U$ be an open set in the interval $[a,b]$.

a. Show that the distance of any point $x$ in $U$ from the complement $U^c = [a,b] \setminus U$, given by $d(x,U^c) = \inf_{y \in U^c} d(x,y)$, is a continuous function on $U$. (Hint: $U^c$ is closed.)

b. Show that the characteristic function $\chi_U$ of an open set $U \subset [a,b]$ is the (pointwise) limit of an increasing sequence of continuous functions. Here, $\chi_U(x) = 1$ if and only if $x \in U$ and otherwise $\chi_U(x) = 0$. Hint: Use the distance function to construct such a sequence, starting with $f_1(x) = \min\{1, d(x,U^c)\}$.

c. Use a result from class to deduce that the increasing sequence is Cauchy in $L^1([a,b])$ and hence that the characteristic function of any open set $U \subset [a,b]$ is in $L^1([a,b])$.

**Problem 4**

Define a map $T : C([0,1]) \to C([0,1])$ by

$$ Tf(x) = \int_0^1 k(x,y)f(y)dy $$

where $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Show that the operator norm $\|T\|$ equals

$$ \|T\| = \sup\{\|Tf\|_\infty : f \in C([0,1]), \|f\|_\infty \leq 1\} = \max_{0 \leq x \leq 1} \int_0^1 |k(x,y)|dy. $$

Hint: If for $x \in [0,1]$, $s(y) = 1$ if $k(x,y) > 0$, $s(y) = 0$ if $k(x,y) = 0$, and $s(y) = -1$ otherwise, then $s$ is in $L^1([0,1])$. Extending the integral to $L^1([0,1])$ then gives $\int_{[0,1]} k(x,y)s(y)dy = \int_0^1 |k(x,y)|dy.$