Assignment 9, due Thursday, December 2, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let \( F : C([-1, 1]) \to \mathbb{R} \) be given by \( F(f) = \int_0^1 f(t) \, dt - \int_{-1}^0 f(t) \, dt \) where \( C([-1, 1]) \) is equipped with \( d_\infty \). Let \( Y = \ker F = \{ f \in C([-1, 1]), F(f) = 0 \} \) and \( h(x) = x \), then show that \( \inf_{y \in Y} \| y - h \| = \frac{1}{2} \) but that there is no \( z \in Y \) with \( \| z - h \| = \frac{1}{2} \). Hint: How does \( |F(f)|/\|F\| \) relate to the distance between \( f \) and \( Y \)?

Problem 2

Let \( 1 < p, q < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \). For \( g \in L^q([a, b]) \), let \( T_g \in (L^p([a, b]))^* \) be given by \( T_g f = \int_{[a, b]} fg \, dx \). Show that \( T : g \mapsto T_g \) is an isometry from \( L^q([a, b]) \) to \( (L^p([a, b]))^* \). Hint: It is enough to show this for a dense set, say for all \( g \in C([a, b]) \). You may quote Hölder’s inequality for \( L^p \)-spaces without proof.

Problem 3

Let for \( j \in \mathbb{N}, a < b, x_0 = a, x_1 = a + (b - a)/2^j, \ldots, x_m = a + (b - a)m/2^j, \ldots, x_{2^j} = b \). Let for \( f \in L^p([a, b]) \), \( T_j f(x) = 2^j \int_{[x_k, x_{k+1}]} f \, dx \) where \( x_k \leq x < x_{k+1} \), then as shown in class, \( \|T_j f\|_p \leq \|f\|_p \) and as \( j \to \infty \), \( T_j f \to f \). Show that for \( F \in (L^p([a, b]))^* \), each \( F_j : f \mapsto F(T_j f) \) is a bounded linear functional on \( L^p([a, b]) \) and as \( j \to \infty \), \( F_j \to F \). Hint: It is enough to show the convergence \( F_j f \to F f \) for \( f \) from a dense set.

For the next problem, you may quote the following fact (weak sequential compactness of the closed ball in \( L^q([a, b]) \)): Let \( 1 < p, q < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( (f_j)_{j=1}^\infty \) is a bounded sequence in \( L^q([a, b]) \), so there is \( M > 0 \) such that for each \( j \in \mathbb{N}, \|f_j\|_q \leq M \), then there is a subsequence \( (f_{j_n})_{n=1}^\infty \) and \( f \in L^q([a, b]) \) such that for each \( g \in L^p([a, b]) \), as \( n \to \infty \), \( \int_{[a, b]} g f_{j_n} \, dx \to \int_{[a, b]} g f \, dx \).

Problem 4

Show that the isometry \( T : g \mapsto T_g \) from \( L^q([a, b]) \) to \( (L^p([a, b]))^* \) is onto.