Assignment 3, due Friday, February 8, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1
Show that that the polynomials $p_0(x) = 1/\sqrt{2}$ and for $n \in \mathbb{N}$
$$p_n(x) = \frac{1}{2^{n+1/2}} \frac{\sqrt{2n + 1}}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$
are what one would obtain from applying the Gram-Schmidt orthonormalization procedure to $1, x, x^2, \ldots$ in $L^2([-1, 1])$. Explain why this is enough to conclude that $\{p_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2([-1, 1])$. You may use without proof that
$$\int_{-1}^{1} (x^2 - 1)^n dx = \frac{(-1)^n (n + 1) 4^{n+1} (n!)^2}{2(n+1)!}.$$

Problem 2
Prove that if $\alpha/2\pi$ is irrational, then for any $2\pi$-periodic continuous function $f$ on $\mathbb{R},$
$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

Hint: First show this for the special case $f(x) = e^{imx}, m \in \mathbb{Z}$.

Problem 3
Prove that a bounded sequence $(x^{(n)})_{n=1}^\infty$ in $\ell^2$ is weakly convergent to $x \in \ell^2$ if and only if for each $k \in \mathbb{N}$, the sequence of $k$-th entries $(x_k^{(n)})_{n=1}^\infty$ converges to $x_k$. Hint: Use that any bounded sequence has a subsequence that is weakly convergent.

Problem 4
Let $A$ be a closed, bounded and convex set in a Hilbert space $H$ and $f : H \to \mathbb{R}$ continuous, convex and bounded below on $A$, so $\inf_{x \in A} f(x) \in \mathbb{R}$. Show that $f$ assumes its minimum on $A$. 