MATH 6361

Applied Analysis

Spring 2019

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 7, due Friday, April 5, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that if T is a compact operator on a complex, infinite-dimensional Hilbert space H and $\langle Tx, x\rangle \geq 0$ for each $x \in H$, then it is self-adjoint and has only non-negative eigenvalues $\lambda_n \geq 0$. With the corresponding orthonormal basis of eigenvectors $(x_n)_{n=1}^\infty$, define $Sy = \sum_{j=1}^\infty \sqrt{\lambda_n}\langle y, x_n\rangle x_n$. Show that S is a well-defined operator on H that is self-adjoint, compact and also has only non-negative eigenvalues, and satisfies $S^2 = T$.

Problem 2

Show that if an operator U on a Hilbert space is unitary, so $UU^* = U^*U = I$ and $U - I$ is compact, then U has a basis of eigenvectors with corresponding eigenvalues $\lambda_n \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Problem 3

Show that the operator V on $L^2([0,1])$ defined by

$$Vf(x) = \int_0^x f(t)dt$$

is compact. Recall the compactness of equicontinuous, closed and bounded families in $C([0,1])$ with respect to the sup-norm.

Problem 4

Let $K(s,t) = 4\cos(s-t)$ and T the extension of

$$Tf(t) = \int_{[-\pi,\pi]} K(t,s)f(s)ds$$

from $f \in C([-\pi,\pi])$ to all of $L^2([-\pi,\pi])$. Find all the eigenvalues and a basis of eigenvectors for T.

Problem 5

Show that if T is normal and compact on H, then it has an eigenvalue $\lambda \in \mathbb{C}$ such that $\|T\| = |\lambda|$. Hint: If T has eigenvalue λ, then T^*T has eigenvalue $|\lambda|^2$.