Assignment 1, due Thursday, January 27, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1
Establish the identity
\[ \|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\|z - \frac{1}{2}(x + y)\|^2 \]
for \(x, y, z\) in a real or complex inner product space.

Problem 2
Let \(P\) be a bounded linear map on a real or complex Hilbert space \(H\). Show that if \(P^2 = P\) (meaning \(P(Px) = Px\) for each \(x \in H\)) and for each \(x, y \in H\), \(\langle Px, y \rangle = \langle x, Py \rangle\), then the range of \(P\) is closed and \(P\) is the orthogonal projection onto its range. Hint: If \(Px = 0\) then \(x \perp \text{ran}(P)\).

Problem 3
Let \(K\) be a closed, convex set in a real Hilbert space \(H\) and \(x_0 \in H \setminus K\). Recall that there is a unique \(y_0 \in K\) such that \(\|y_0 - x_0\| = \inf_{y \in K} \|y - x_0\|\). Show that \(z \in K\) satisfies \(z = y_0\) if and only if for each \(y \in K\), \(\langle x_0 - z, y \rangle \leq \langle x_0 - z, z \rangle\). Hint: Consider for \(t \in [0, 1]\) the squared distance between \(x_0\) and \(ty_0 + (1-t)z\).

Problem 4
The complex exponentials \(u_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}\), define an orthonormal basis for \(L^2([-\pi, \pi])\), so for \(f \in L^2([-\pi, \pi])\), \(\|f\|^2 = \sum_{n=-\infty}^{\infty} |\langle f, u_n \rangle|^2\). Evaluate both sides of this identity for (a) \(f(x) = 1\), if \(x \geq 0\) and \(f(x) = 0\), if \(x < 0\) and (b) \(f(x) = \pi - |x|\). On one side, you get a number, on the other, a series.