Assignment 9, due Thursday, April 28, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let \( H \) be a real Hilbert space and \( K \) be a non-empty, convex, closed and bounded set and \( x \not\in K \). Show that there exists a bounded linear functional \( f \) such that \( \inf_{y \in K} f(y) > f(x) \). Hint: First treat the special case \( x = 0 \). Recall that there is an element in \( K \) which minimizes the norm.

Problem 2

Consider \( \ell^2 \) as a real Hilbert space, containing each square-summable sequence \( x = (x_1, x_2, \ldots) \). Consider the sets

\[
A = \{ x \in \ell^2 : k|x_k - k^{-2/3}| \leq x_1 \text{ for each } k \in \mathbb{N} \}
\]

and

\[
B = \{ x \in \ell^2 : x_k = 0 \text{ if } k \geq 2 \}.
\]

a. Prove that \( A \) and \( B \) are closed convex sets and that \( A \cap B = \emptyset \).

b. Show that \( A - B = \{ x \in \ell^2 : \text{there is } C \geq 0 \text{ such that } k|x_k - k^{-2/3}| \leq C \text{ for each } k \geq 2 \} \).

c. Use the preceding result to show that \( A - B \) is dense in \( \ell^2 \).

d. Prove that \( A \) and \( B \) cannot be separated by a bounded linear functional.

Problem 3

Let \( \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R} \), and let \( \epsilon = \min \{|x_i - x_j| : i \neq j\} \). Suppose that there is a function \( F : \mathbb{R} \to \{+1, -1\} \) that is onto. For \( \lambda > 0 \) and each \( j \in \{1, 2, \ldots, n\} \), define \( f_j \in L^2(\mathbb{R}) \) by

\[
f_j(y) = \frac{1}{2\lambda} e^{-|y-x_j|/\lambda}.
\]

Consider for each \( x_j \) the half-open interval \( I_j = [x_j - \epsilon/2, x_j + \epsilon/2) \), and form the linear combination of characteristic/indicator functions of these half-open intervals

\[
g(x) = \sum_{j=1}^{n} F(x_j) \chi_{I_j}(x),
\]

which defines a bounded linear functional \( G \) on \( L^2(\mathbb{R}) \) by \( G(f) = \int_{\mathbb{R}} f(x)g(x)dx \). Show that if \( \lambda < \epsilon/(2 \ln 2) \), then for each \( j \), \( G(f_j) > 0 \) if \( F(x_j) = 1 \) and \( G(f_j) < 0 \) if \( F(x_j) = -1 \).