Stochastic Processes - Spring 2008

Practice Problems for Final Exam
Bernhard Bodmann, PGH 636
Duration: 150 minutes

Show all work. No points will be given for numerical answers without working being shown.

(1) Consider the (continuous-time) Poisson process \( \{N_t\}_{t \geq 0} \), which has independent increments on disjoint intervals, with distribution given by

\[
P(N_t - N_s = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}
\]

for all \( t \geq s \geq 0, k \in \{0, 1, 2, \ldots\} \) and a fixed parameter \( \lambda > 0 \). Show that the process \( M_t = N_t - \lambda t \) is a martingale.

Solution Since by the triangle inequality \( \mathbb{E}[|M_t|] \leq \mathbb{E}[|N_t|] + \lambda t \leq \infty \) and increments are independent, we only need to check that the expected value is constant:

\[
\mathbb{E}[M_t] = \mathbb{E}[N_t] - \lambda t = 0.
\]

The last equality follows from the definition of the Poisson distribution and from \( \mathbb{E}[N_t] = \mathbb{E}[N_t - N_0] = \lambda t. \)

(2) Suppose \( \{B_t\}_{t \geq 0} \) is a standard Brownian motion starting at \( B_0 = 0 \). Let \( \tilde{B}_t = tB_1 - B_t. \)

(a) Compute \( \mathbb{E}[\tilde{B}_sB_t] \). Give a reason why \( B_t \) is independent of \( \sigma(\{\tilde{B}_s : 0 \leq s \leq 1\}) \).
Solution. The new process is a linear combination of Gaussian processes, thus again Gaussian. By definition $E[\tilde{B}_t] = E[tB_1 - B_1] = 0$.

Computing the covariance gives

$$E[\tilde{B}_sB_1] = E[(sB_1 - B_s)B_1] = sE[B_1B_1] - E[B_sB_1] = s - s = 0.$$  

Thus, $\tilde{B}_s$ and $B_1$ are uncorrelated, and since they are jointly Gaussian, independent. A similar argument works when $\tilde{B}_s$ is replaced by a vector $\{\tilde{B}_{s_j}\}_{j=1}^n$ with $0 \leq s_1 < s_2 < \cdots < s_n < 1$.

(b) Compare for fixed $s \in [0, 1]$ the distributions of $\tilde{B}_s$ and $\tilde{B}_{1-s}$.

Solution. Again, since both $\tilde{B}_s$ and $\tilde{B}_{1-s}$ are Gaussian, we only need to compare expectation values and their variances. We obtain $E[\tilde{B}_s] = E[\tilde{B}_{1-s}] = 0$ and

$$E[\tilde{B}_s^2] = s^2E[B_1^2] - 2sE[B_1B_s] + E[B_s^2] = s - s^2.$$  

But we have $g(s) = s - s^2 = g(1 - s)$, so the variance is the same for $\tilde{B}_{1-s}$.
Let $W_t$ be a Brownian motion with drift parameter $\mu$, that is $W_t = B_t + \mu t$.

(a) Show that for any real $\lambda > 0$

$$V_t = e^{\lambda W_t - (\lambda \mu + \frac{\lambda^2}{2}) t}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$.

Solution This is the same as showing $e^{\lambda B_t - \lambda^2 t/2}$ is a martingale. We check the martingale properties directly. Integrability is straightforward by the Gaussian decay of the density of $B_t$. Consider the canonical filtration $\mathcal{F}_s$.

We have

$$E[e^{\lambda B_t - \lambda^2 t/2} | \mathcal{F}_s] = E[e^{\lambda B_t + \lambda (B_t - B_s) - \lambda^2 (t-s)/2 + \lambda^2 s/2} | \mathcal{F}_s]$$

$$= e^{\lambda B_s - \lambda^2 s/2} E[e^{\lambda (B_t - B_s) - \lambda^2 (t-s)/2} | \mathcal{F}_s]$$

In the last step we have extracted the $\mathcal{F}_s$-measurable factor and then used that the increment $B_t - B_s$ is independent of $\mathcal{F}_s$. Now we see

$$E[e^{\lambda (B_t - B_s) - \lambda^2 (t-s)/2}] = \frac{1}{\sqrt{2 \pi u}} \int_{-\infty}^{\infty} e^{\lambda x - \lambda^2 u/2} e^{-x^2/2u} dx$$

$$= \frac{1}{\sqrt{2 \pi u}} \int_{-\infty}^{\infty} e^{-((x/\sqrt{u} - \lambda \sqrt{u})^2)} dx = 1.$$ 

(b) Taking $\lambda = -2\mu$ in (a) you may conclude that $V_t = e^{-2\mu W_t}$ is a martingale. By using a stopping time argument or otherwise show that the probability that the Brownian motion with drift $\mu$ reaches $b > 0$ before $a < 0$ is

$$1 - e^{-2\mu a}$$

$$e^{-2\mu b} - e^{-2\mu a}.$$ 

Solution We assume that the conditions of Doob’s optional stopping theorem hold, so

$$E[e^{-2\mu W_T}] = E[e^0] = 1$$

but, calling $P_b$ the desired probability, this expected value is

$$E[e^{-2\mu W_T}] = P_b e^{-2\mu} + P_a e^{-2\mu a}.$$
Now using $P_a = 1 - P_b$ and solving for $P_b$ gives the desired result.

\textbf{(4) (a)} Let $B_t$ denote standard Brownian motion and $\mathcal{F}_t$ the $\sigma$-algebra generated by the random variables $\{B_s\}, 0 \leq s \leq t$.

Let $Y_t = \max_{0 \leq s \leq t} B_s$. Use the reflection principle to show that for all $t \geq 0$, the distribution of $Y_t$ is identical to that of $|B_t|$.

\textit{Solution} The reflection principle shows

$$\mathbb{P}(\max_{0 \leq s \leq t} B_s \geq \alpha) = 2\mathbb{P}(B_t \geq \alpha)$$

but by the symmetry of Brownian motion this is $2\mathbb{P}(B_t \geq \alpha) = \mathbb{P}(|B_t| \geq \alpha)$.

\textbf{(b)} Let for fixed $t \geq 0$, the family of random variables $\{S_h\}_{0 < h < 1}$ be

$$S_h = \frac{B_{t+h} - B_t}{h}.$$  

Show this family has diverging norm in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$\sup_{0 < s < 1} \mathbb{E}[S_h^2] = \infty.$$  

\textit{Solution} The squared norm of $S_h$ is by definition

$$\mathbb{E}[(B_{t+h} - B_t)^2] = \frac{1}{h^2} \mathbb{E}[(B_{t+h} - B_t)^2] = \frac{1}{h}$$

where we have used that the variance of the increments of BM is equal to the time difference. This last quantity is clearly unbounded.
(5) Suppose that \( \{X_n\} \) is a Markov chain with countable state space \( S = \mathbb{N} \) and transition probability matrix \( P = (P_{ij}) \). Suppose \( (V_i) \) is a right eigenvector for \( P \) with eigenvalue \( \lambda \) i.e. for all \( i \in \mathbb{N} \),
\[
\sum_j P_{ij} V_j = \lambda V_i
\]
such that \( \mathbb{E}[|V_{X_n}|] < \infty \) for all \( n \). Show that
\[
Y_n = \frac{V_{X_n}}{\lambda^n}
\]
is a martingale with respect to the filtration \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \)

**Solution.**

To test the martingale property, we have to verify that given \( X_n = k \), then the expected value of \( V_{X_{n+1}}/\lambda^{n+1} \) is \( V_k/\lambda^n \).

We know the distribution of \( X_{n+1} \) is given by
\[
P(X_{n+1} = i|X_n = k) = P_{ki},
\]
so the expected value is by the eigenvalue equation
\[
\sum_i P_{ki} V_i/\lambda^{n+1} = V_k/\lambda^n.
\]

(6) Let \( Z_n \) be the population for the \( n \)-th generation of the branching process for which each node numbered \( i = 1, 2, \ldots Z_n \) independently branches into \( X_i^{(n)} \) nodes at the following generation, with mean \( \mathbb{E}[X_i^{(n)}] = m > 1 \) and variance \( \sigma^2 = \text{Var}[X_i^{(n)}] \). Let \( Z_0 = 1 \).

(a) Compute \( \mathbb{E}[Z_n] \). (Hint: Use a conditional expectation to relate \( \mathbb{E}[Z_n] \) and \( \mathbb{E}[Z_{n+1}] \).

**Solution.**

Using conditioning on \( Z_n \) and \( Z_{n+1} = X_1^{(n)} + \ldots X_{Z_n}^{(n)} \), we have
\[
\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1}|Z_n]] = m\mathbb{E}[Z_n].
\]
Therefore, \( \mathbb{E}[Z_n] = m^n \).
(b) Verify that \( \mathbb{E}[Z_{n+1}^2] = m^2 \mathbb{E}[Z_n^2] + \sigma^2 \mathbb{E}[Z_n] \).

**Solution.**

From the recursion relation for the generating function, \( f_n(s) = \mathbb{E}[s^{Z_n}] \), which is \( f_{n+1}(s) = f_n(f(s)) \), we obtain by differentiating twice

\[
\frac{d^2}{ds^2} f_{n+1}(s) = f_n''(f(s))(f'(s))^2 + f_n'(f(s))f''(s)
\]

now setting \( s = 1 \) gives

\[
\mathbb{E}[Z_{n+1}^2 - Z_n] = (\mathbb{E}[Z_n^2] - \mathbb{E}[Z_n])(\mathbb{E}[X])^2 + \mathbb{E}[Z_n]\mathbb{E}[X^2 - X]
\]
\[
= m^2(\mathbb{E}[Z_n^2] - \mathbb{E}[Z_n]) + \mathbb{E}[Z_n](m^2 + \sigma^2 - m)
\]
\[
= m^2 \mathbb{E}[Z_n]^2 + (\sigma^2 - m) \mathbb{E}[Z_n].
\]

Now adding the expected value \( \mathbb{E}[Z_{n+1}] = m \mathbb{E}[Z_n] \) to both sides gives the desired identity.