Information Theory with Applications

MATH 6397 – Fall 2014

September 4, 2014

Homework Set 1, due Tuesday, Sep 16, 2014

1. **Additivity of entropy.** Let X, Y, Z be three binary random variables. Assume the random vector (X, Y, Z) has equal probability $1/4$ for each outcome $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$ and $(1, 0, 1)$.

   (a) Write down six ways of splitting $H(X, Y, Z)$ into three terms involving conditional entropy.

   (b) Compute $H(X)$, $H(Y|X)$ and $H(Z|X, Y)$.

   (c) Conclude the value of $H(X, Y, Z)$ and verify this result by a direct way of computing $H(X, Y, Z)$.

2. **Conditional entropy as distance?** Given three random variables X, Y, Z which map to an at most countable alphabet, and the usual definition of conditional entropy, show the triangle-type inequality

$$H(X|Z) \leq H(X|Y) + H(Y|Z) .$$

3. **Entropy and counting.** Let $\alpha \leq 1/2$, and let $h(\alpha)$ be the binary entropy (the entropy of a binary random variable with success probability α). Show that for each $n \in \mathbb{N}$, if $I = \{i \in \mathbb{Z} : 0 \leq i \leq \alpha n\}$, then

$$\sum_{i \in I} \binom{n}{i} \leq e^{nh(\alpha)} .$$

   Hints: Recall that $\binom{n}{i}$ counts the number of binary sequences of length n with i ones. Consider a sequence of n random variables $\{X_1, X_2, \ldots, X_n\}$ which produce each binary sequence with at most $n\alpha$ ones with equal probability. Use additivity and entropy inequalities to deduce the claimed upper bound.