Homework Set 3, due Tuesday, Nov 11, 2014

1. Optimal quantization. Let $X$ be a random variable which is normally distributed with mean zero and variance $\sigma^2$. Which map $\phi : \mathbb{R} \to \{a, b\}$ and choice of the two numbers $a < b$ minimizes the mean-square error $\mathbb{E}[(X - \phi(X))^2]$?

2. Entropy maximizers. Prove that among all non-negative random variables with a given mean $\mu > 0$, the exponential distribution has maximal differential entropy.

3. Rate distortion theory for binary random variables.
   (a) Let $X$ be a random variable with values in $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X = 0) = p < 1/2$. Assume $0 \leq \epsilon \leq p$. Show that for any random variable $Y$ with the same alphabet and $\mathbb{P}(X \neq Y) \leq \epsilon$, we have
   $$I(X; Y) \geq h(p) - h(\epsilon)$$
   where $h$ is the binary entropy.
   (b) Let $X$ be as before, and assume $0 \leq \epsilon \leq p < 1/2$. Define a random variable $Y$ such that it has the marginal distribution satisfying
   $$\mathbb{P}(Y = 0) = \frac{1 - p - \epsilon}{1 - 2\epsilon},$$
   and conditioning on the outcome of $Y$ gives $\mathbb{P}(X = 0|Y = 0) = 1 - \epsilon$, $\mathbb{P}(X = 0|Y = 1) = \epsilon$. Prove that
   $$I(X; Y) = h(p) - h(\epsilon).$$
   (c) Given a discrete memoryless source $\{X_j\}_{j=1}^{\infty}$ with alphabet $\mathbb{A} = \{0, 1\}$ and $\mathbb{P}(X_1 = 0) = p < 1/2$, use the preceding parts to show
that for the additive Hamming distortion ($d_n$ counts the number of errors for binary input/output sequences of length $n$),

$$R(D) = \begin{cases} h(p) - h(D), & \text{if } 0 \leq D < p \\ 0, & \text{if } D > p \end{cases}.$$