Information Theory with Applications, Math6397
Lecture Notes from September 23, 2014

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Since the AEP holds, we have block coding for the stationary ergodic case as for DMS.

2.3.12 Theorem. *Let $\{X_j\}_{j=-\infty}^{\infty}$ be a stationary ergodic source and $H_\infty = \lim_{n\to\infty} H(X_n|X_1, X_2, X_3, \ldots, X_n)$ then for every $\epsilon > 0$ there is δ, $0 < \delta < \epsilon$ and a sequence of codes. $\{(\mathcal{C}_n, \phi_n)\}_{n=1}^{\infty}$ with coding rate $R = \lim_{n\to\infty} \frac{1}{n} \ln |\mathcal{C}_n| < H_\infty + \delta$ such that for all sufficiently large n, $\mathbb{P}$(decoding error) $< \epsilon$*

Proof. As before, since you only use AEP. □

We also have the converse of block coding:

2.3.13 Theorem. *Let $\{(\mathcal{C}_n, \phi_n)\}_{n=1}^{\infty}$ be a sequence of block codes with $R = \limsup_n \frac{1}{n} \ln |\mathcal{C}_n| <$ then for all $\lambda > 0$ and any choice $\{\psi_n\}_{n=1}^{\infty}$ if n is sufficiently large,*
we have P(decoding error) $> 1 - \lambda$.

Proof. As before. □

2.3.14 Question. There are instances in which we would like to have no mistakes, a loss-less situation. How can we get codes such that

$$\mathbb{P}(\text{decoding error}) \xrightarrow{n\to\infty} 0 \text{ or even } \mathbb{P}(\text{decoding error}) = 0?$$

2.3.15 Answer. Allow an infinite size code book

2.4 Separable Codes and Prefix Codes

Block coding in reality uses a sequence $x \in \mathbb{A}^n$ that is mapped to $\mathbb{Q}_n(x) \in \mathbb{B}^\ell$ for a code alphabet $\mathbb{B}$ with size $|\mathbb{B}| = \mathsf{K}$. You can think of K as the base of a number system, which motivates calling this K-**ary encoding**.

When the length of the code sequence is no longer fixed, we speak of *fixed-variable* coding, with the codebook a subset of $\bigcup_{\ell=1}^{\infty} \mathbb{B}^\ell$

2.4.16 Definition. A map ϕ with range $\mathcal{C} \subset \bigcup_{\ell=1}^{\infty} \mathbb{B}^\ell$ is called **regular** if it is $1 - 1$.

Usually we need to encode sequences $\{x_1, x_2, \ldots, x_n\}$, with $x_j \in \mathbb{A}$.

2.4.17 Definition. A code $(\mathcal{C}, \phi)$ is called **separable** if we can extend the $1-1$ map ϕ to sequences by concatenation,

$$\phi(\{x_1, x_2, \ldots, x_m\}) = \{\phi(x_1), \phi(x_2), \ldots, \phi(x_m)\}$$

and this concatenation map is invertible for all $m \in \mathbb{N}$.

So if $\mathbb{A} = \{$A,B,C,D,E,F$\}$, $\mathbb{B} = \{0, 1\}$ and we chose block length 1 for $\mathbb{A}$, we could encode in the way described in Table 2 to achieve a separable code.

$x \in \mathbb{A}$	$\phi(x)$
A	0
B	10
C	110
D	1110
E	11110
F	111110

Table 2: Example of a separable binary code for source alphabet $\{A, B, C, D, E, F\}$.

Usually, we need to read the entire message to separate words, however our example shows that there is a method that allows for iterative decoding. These are called **prefix codes**.

2.4.18 Definition. A code $\phi : \mathbb{A} \to \mathcal{C} = \bigcup_{\ell=1}^{\infty} \mathbb{B}^\ell$ is called a **prefix code** if no code-word is the prefix of another.

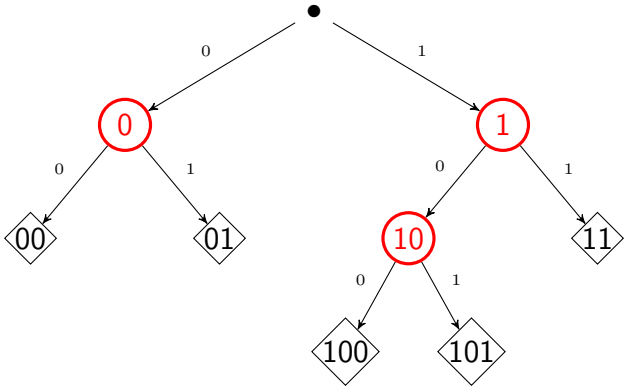
2.4.19 Example. Prefixes of $\{b_1, b_2, \ldots, b_\ell\} \in \mathbb{B}^\ell$ are $\{b_1\}, \{b_1 b_2\}, \{b_1 b_2 b_3\} \ldots, \{b_1 b_2, \ldots, b_\ell\}$

2.4.20 Example. Examples of fixed-variable codes

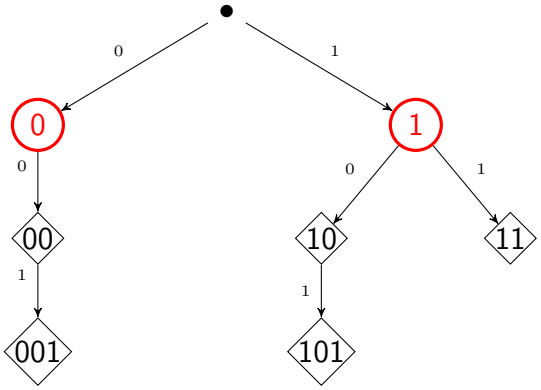
- $\mathcal{C}_1 = \{\{0,0\}\{0,1\}\{1,1\}\{1,0,0\}\{1,0,1\}\}$
- $\mathcal{C}_2 = \{\{0,0\}\{1,0\}\{1,1\}\{0,0,1\}\{1,0,1\}\}$
- $\mathcal{C}_3 = \{\{0,0\}\{0,1\}\{1,1\}\{1,0,0\}\{1,1,0\}\}$

2.4.21 Question. Which of these is prefix / separable? Is there a systematic way to find out if something is a prefix code or separable?

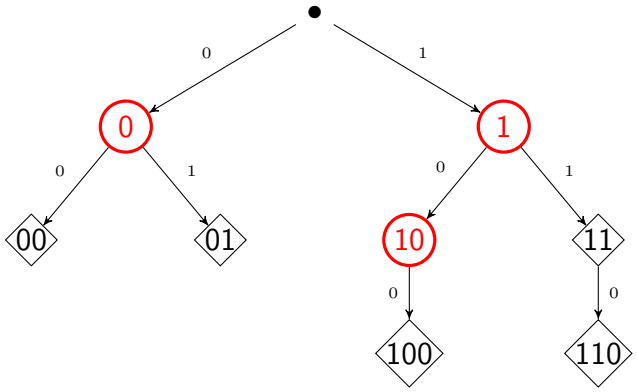
2.4.22 Answer. We can build a tree!



(a) $\mathcal{C}_1 = \{\{0,0\}\{0,1\}\{1,1\}\{1,0,0\}\{1,0,1\}\}$.
$\mathcal{C}_1$ is a valid prefix code.



(b) $\mathcal{C}_2 = \{\{0,0\}\{1,0\}\{1,1\}\{0,0,1\}\{1,0,1\}\}$.
$\mathcal{C}_2$ is a not a prefix code, but it is separable because it is a backward prefix.



(c) $\mathcal{C}_3 = \{\{0,0\}\{0,1\}\{1,1\}\{1,0,0\}\{1,1,0\}\}$.
$\mathcal{C}_3$ we can see is neither prefix or separable with the sequence $\{1,1,0,0,0,1,0,0\}$ could be interpreted as $\{\{1,1,0\},\{0,1\},\{0,0\}\}or\{\{1,1\},\{0,0\},\{1,0,0\}\ \}$

2.4.23 Remark. We note that the K-ary prefix codes are characterized by a K-ary tree with **nodes** (vertices with ≥ 2 adjacent nodes) and **leaves** (vertices with only one edge).

Decoding is a simple iterative procedure. While reading along the coded sequence and following the tree, if we arrive at a leaf (diamond symbol) we record the coded symbol and start again from the root.

2.4.24 Question. How long do codewords have to be?

2.4.25 Theorem. *Given $|\mathbb{B}| = k$, then for any separable code $\phi : \mathbb{A} \to \bigcup_{\ell=1}^{\infty} \mathbb{B}^\ell$,*

$$\sum_{x \in \mathbb{A}} K^{-\ell(x)} \leq 1\,.$$

Here $\ell(x)$ is the length of $\phi(x)$ and the size of $\mathbb{A}$ is implicit as you are summing over all elements.

Proof. Assigning $n\ell_{\mathsf{max}}$ as the maximal length of $\phi(x), x \in \mathbb{A}$, Consider the n-th power of the

left-hand side

$$\begin{aligned}(\sum_{x\in\mathbb{A}} K^{-\ell(x)})^n &= \sum_{x_1\in\mathbb{A}}\sum_{x_2\in\mathbb{A}}\cdots\sum_{x_n\in\mathbb{A}} K^{-\ell(x_1)-\ell(x_2)\cdots-\ell x_n}\\ &= \sum_{(x_1,x_2,\ldots,x_n)\in\mathbb{A}^n} K^{-\ell(x_1,x_2,\ldots,x_n)}\\ &\leq \sum_{m=1}^{n\ell_{\max}} \underbrace{A(m)}_{\substack{\text{number of codewords}\\ \text{of size } A(m)\leq K^m}} K^{-m}\\ &\leq n\ell_{\max}\end{aligned}$$

so taking the n-th root on both sides and n$\to\infty$, we get $\sum_{x\in\mathbb{A}} K^{-\ell(x)} \leq (n\ell_{\max})^{\frac{1}{n}} \xrightarrow{n\to\infty} 1$ □

This is often called the **Kraft inequality** and it tells you how much flexibility you have.

2.4.26 Question. How long is a codeword on average?

2.4.27 Theorem. *Given a DMS with values in $\mathbb{A}$ and induced measure $\mathbb{Q}$ on $\mathbb{A}$ for all $j \in \{1, 2, \dots\}$ if code is separable*

$$\mathbb{E}[\ell(X_j)] \geq \underbrace{H_K(\mathbb{Q})}_{K\text{-ary entropy, with } \log \text{ of base } K \text{ instead of } \ln}.$$

Which of the choices that the Kraft inequality gave you will let you get close to the bound? Huffman coding is the answer.