Last Time

- Linear codes for continuous channels.

Summary of the last result:

**Theorem:** Given a random input vector: \( X : \Omega \rightarrow \mathbb{R}^k \), \( X_j \) i.i.d. with zero mean, Gaussian components having covariance matrix: \( C_X = \frac{S}{k} I \) and a channel \( \gamma : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n \), \( \gamma(\hat{X}) = \hat{X} + N \) then the best linear encoding achieves:

\[
C(S) = \sum_{j=1}^{k} \frac{1}{2} \ln \left( \frac{1}{\mu \sigma_j^2} - \frac{k}{S} \right)^+ + 1
\]

where \( \{\sigma_j^2\}_{j=1}^{k} \) are the k smallest eigenvalues of \( C_N \) and \( \mu \) is chosen such that:

\[
\sum_{j=1}^{k} \left( \frac{1}{\mu k} - \frac{\sigma_j^2}{S} \right)^+ = 1
\]

where \( \frac{1}{\sigma_j^2} = \lambda_j \)

**Note:** If noise has components, \( \sigma_j = \sigma \) then using a linear encoding into \( \mathbb{R}^n \) with large 'n' is of no use.


8 Frames as Codes

Recall: A q-ary block code of length $m$, $\phi_n : \mathbb{A}^k \rightarrow \mathbb{A}^n$, $|A| = q$ has rate:

$$R = \frac{\log_q(m)}{n} = \frac{\log_q|A|^k}{n} = \frac{\log_q(q)^k}{n} = \frac{k}{n}$$

(also known as 'coding rate')

For invertible $\phi_n$ we have : $R \leq 1$ or $k \leq n$.

8.0.1 Definition:

If $\phi_n : F^k \rightarrow F^n$ and $F = \mathbb{R}$ or $F = \mathbb{C}$ then we say that $\phi_n$ has dimensionless rate $R = \frac{k}{n}$. If $\phi_n(a) = Ha$, where $H$ is a $n \times k$ matrix, then we say the code is linear and $H$ is called the encoding matrix.

In this case by linearity of $H$ and Riesz representation theorem (for Hilbert spaces):

$$(Ha)_j = \langle a, f_j \rangle$$

and if $H$ is left-invertible, then there exists $c > 0$ such that: $H^*H \geq cI$ where $H$ is a positive definite matrix and we have an operator inequality, i.e.

$$\langle H^*Hx, x \rangle = ||Hx||^2 \geq c||x||^2$$

where $\sqrt{c}$ is the strictly positive distance of $x$ from 0.

Also by boundedness of $H$, there is $C > 0$, such that : $H^*H \leq C I$. Thus for each $x$ in $F^k$:

$$c||x||^2 \leq ||Hx||^2 = \sum_{j=1}^{n} | \langle x, f_j \rangle |^2 \leq C||x||^2$$

If these inequalities hold for $0 < c \leq C < \infty$, then we say that $\{f_j\}_{j=1}^{n}$ forms a frame for $F^k$.

Remark: Given a frame $\{f_j\}_{j=1}^{n}$ for $F^k$, then : $\sum_{j=1}^{n} \langle x, f_j \rangle f_j = H^*Hx$, where $H^*H$ is an invertible matrix. So we have, for all $x$ in $F^k$:

$$\sum_{j=1}^{n} \langle x, f_j \rangle (H^*H)^{-1}f_j = x$$

8.0.2 Definition:

For $\{f_j\}_{j=1}^{n}$ frame for $F^k$, with associated map : $H : F^k \rightarrow F^n$ we call : $g_j = (H^*H)^{-1}f_j$ the canonical dual frame to $\{f_j\}_{j=1}^{n}$.

We can restore the signal with with a different set of vectors. For example, if $K$ is an $n \times n$-matrix such that $HH^*K = 0$, then

$$\sum_{j=1}^{n} (\delta_{j,l} + K_{j,l}) < x, f_i > (H^*H)^{-1}f_j = x$$

and thus $h_j = \sum_{l=1}^{n}(\delta_{l,j} + K_{l,j})(H^*H)^{-1}f_i$ is another choice for a dual.
8.0.3 Definition:

If a frame is identical to its canonical dual, \( g_j = f_j \) for all \( j \), then we call \( \{f_j\}_{j=1}^n \) a Parseval frame.

In this case, using \( H^*H = I_k \) gives:

\[
\sum_{j=1}^n <x, f_j><f_j, x> = <x, x> = ||x||^2
\]

similar to the Parseval identity for orthonormal basis, though the frame vectors do not have to be orthonormal.

8.0.4 Lemma

If \( \{f_j\}_{j=1}^n \) is a Parseval frame for \( \mathbb{R}^k \) and \( H \) the associated encoding matrix, then among all left inverses of \( H \), \( H^* \) has minimal operator norm and Hilbert-Schmidt norm.

Proof: Let \( G \ (k \times n \text{ matrix}) \) be another left-inverse of \( H \) or \( GH = I_k \).

- For the Operator Norm:

\[
||G|| = \max_{y \in \mathbb{P}^n} \frac{||Gy||}{||y||} \geq \max_{y \in \mathbb{P}^n} \frac{||Gy||}{||y||} \quad \text{(since we maximize over a lesser range now)}
\]

\[
= \max_{x \in \mathbb{P}^k} \frac{||GHx||}{||Hx||} \quad \text{(Here } GH = I \text{ and } ||Hx|| = ||x|| \text{ by Parseval)}
\]

or \( ||G|| \geq 1 \)

Also: \( ||H^*|| = ||H|| = 1 \)

Thus we have: \( ||G|| \geq ||H^*|| \), or \( H^* \) has the minimal operator norm.

- For the Hilbert-Schmidt Norm: For any orthonormal basis \( \{e_i\}_{i=1}^k \) of \( \mathbb{R}^k \), using that \( \{He_i\}_{i=1}^k \) is an orthonormal system (not basis), we get:

\[
tr[G^*G] \geq \sum_{i=1}^k <G^*GH e_i, He_i>
\]

\[
= \sum_{i=1}^k <GH e_i, GH e_i> \quad \text{(using } GH = I \text{)}
\]

\[
= \sum_{i=1}^k <e_i, e_i> \quad \text{(since orthonormal)}
\]

\[
= k = tr[H^*H]
\]

or \( tr[G^*G] \geq tr[HH^*] \)

Thus \( H^* \) is the optimal choice for Hilbert-Schmidt norm.
Parseval frames have an interesting geometric property. Since the encoding matrix $H$ is an isometry, $H^*H = I$, and we obtain the trace identity

$$k = tr[H^*H] = tr[HH^*] = \sum_{j=1}^{n} <f_j, f_j> = \sum_{j=1}^{n} ||f_j||^2.$$  

8.0.5 Corollary:

For a Parseval frame $\{f_j\}_{j=1}^{n}$, $\sum_{j=1}^{n} ||f_j||^2 = k$.

Thus, we can think of $\{f_j\}_{j=1}^{n}$ as being a 'vector-valued' sphere.

8.0.6 Definition:

If a frame $\{f_j\}_{j=1}^{n}$ has only vectors of norm $||f_j|| = c$, then we call it an equal-norm frame.

8.0.7 Corollary:

If $\{f_j\}_{j=1}^{n}$ is an equal-norm and Parseval frame, then $||f_j|| = \sqrt{\frac{k}{n}}$ for each $j$.

Proof: Since $\{f_j\}_{j=1}^{n}$ is an equal-norm frame, using $k = \sum_{j=1}^{n} ||f_j||^2$, we get:

$$k = n||f_j||^2$$

$$\Rightarrow ||f_j|| = \sqrt{\frac{k}{n}}$$

8.0.8 Definition:

We call a Parseval frame $\{f_j\}_{j=1}^{n}$ for $\mathbb{R}^k$ a $(n,k)$-frame.

8.1 Erasures

Given an encoding with the Parseval frame by associated isometry, $V : \mathbb{R}^k \rightarrow \mathbb{R}^n$, consider the "loss" of frame co-efficients. This means, we cannot use the value of certain coefficients to reconstruct or even approximate a vector $x$. To model this in a mathematically concise form, we set the corresponding frame coefficients to zero.
8.1.1 Definition:

A one erasure $E_l$, indexed by $l \in \{1, 2, ..., n\}$ is a map $E_l : \mathbb{R}^k \to \mathbb{R}^n$ given by:

$$
x \mapsto \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & & & & : \\
\vdots & & 1 & & & & : \\
\vdots & & & 0 & (l^{th} \text{ column}) & \ddots & : \\
\vdots & & & & 1 & & : \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix} x
$$

A general erasure is a diagonal projection matrix.

Qs. Can we recover $x$ from $EVx$, where $E$ is a general erasure and how?