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8 Frames for the encoding of analog data

A finite frame $\Phi = \{\varphi_j\}_{j=1}^M$ is a spanning family of vectors in an N-dimensional real or complex Hilbert space $\mathcal{H}$. If the Parseval-type identity

$$\|x\|^2 = \frac{1}{A}\sum_{j=1}^{M} |\langle x, \varphi_j\rangle|^2$$

is true for all all $x \in \mathcal{H}$ with some constant $A > 0$, then Φ is called A-*tight*. If A=1, then we say that Φ is a *Parseval frame*. In this case, we also call it an (M, N)-frame, in analogy with the literature on block codes. The analysis operator of a frame Φ is the map $T : \mathcal{H} \to \ell^2(\{1, 2, \ldots, M\})$, $(Tx)_j = \langle x, \varphi_j\rangle$. If Φ is a Parseval frame, then T is an isometry. Frames are often classified by geometric properties: If all the frame vectors have the same norm then the frame is called *equal-norm*. If the frame is tight and there is $c \geq 0$ such that for all $j \neq l$, $|\langle \varphi_j, \varphi_l\rangle| = c$ then the frame is called *equiangular and tight*. The significance of the geometric characteristics of frames is that they are related to optimality of frame designs in certain situations. This will be reviewed in the following material.

The general model for frame-coded transmissions contains three parts: (1) the linear encoding of a vector in terms of its frame coefficients, (2) the transmission which may alter the frame coefficients and (3) the reconstruction algorithm. The input vectors to be transmitted can either be assumed to have some distribution, or one can attempt to minimize the reconstruction error among all possible inputs of a given norm. The same can be applied to the errors occurring in the transmission. Our discussion restricts the treatment of input vectors to the worst-case scenario, or to the average over the uniform probability distribution among all unit-norm input vectors. The channel models are taken to be either the worst case or a uniform erasure distribution, possibly together with the addition of independently distributed random components to the frame coefficients which model the digitization noise for the input. Among all possible reconstruction algorithms, we concentrate on linear ones, which may or may not depend on the type of error that occurred in the course of the transmission.

8.1 Frames for erasure channels

A standard assumption in network models is that a *sequence* of vectors is transmitted in the form of their frame coefficients. These coefficients are sent in parallel streams to the receiver, see [7, Example 1.1] and [12]. If one of the nodes in the network experiences a buffer overflow or a wireless outage, then the streams passing through this node are corrupted. The integrity of each coefficient in a transmission is typically protected by some error correction scheme, so for practical purposes one may assume that coefficients passing through the affected node are not used in the reconstruction process. The linear reconstruction of a vector from a subset of its frame coefficients amounts to setting the lost coefficients to zero, a so-called erasure error. This type of error has been the subject of many works [8, 7, 6, 12, 10, 2]. In our formulation, the encoded vector is given by its frame coefficients Tx, and the erasure acts by applying a diagonal projection matrix E to Tx, before linear reconstruction is attempted.

We can either reconstruct by an erasure-dependent linear transform, performing active error correction, or use so-called blind reconstruction, which ignores the fact some coefficients have been set to zero. For now, we focus on the second alternative and add some comments about active error correction later.

8.1.1 Definition. Let Φ be an (M, N)-frame for a real or complex Hilbert space $\mathcal{H}$, with analysis operator T. The *blind reconstruction error* for an input vector $x \in \mathcal{H}$ and an erasure of frame coefficients with indices $\mathbb{K} = \{j_1, j_2, \dots j_m\} \subset \mathbb{J} = \{1, 2, \dots, M\}$, $m \leq M$, is given by

$$\|T^* E_{\mathbb{K}} T x - x\| = \|(T^* E_{\mathbb{K}} T - I)x\| = \|T^*(I - E_{\mathbb{K}})Tx\|$$

where E is the diagonal $M \times M$ matrix with $E_{j,j} = 0$ if $j \in \mathbb{K}$ and $E_{j,j} = 1$ otherwise. The residual error after performing active error correction is defined as $\|W E_{\mathbb{K}} T x - x\|$, where W is the Moore-Penrose pseudoinverse of $E_{\mathbb{K}} T$. If $W E_{\mathbb{K}} T = I$, then we say that the erasure of coefficients indexed by $\mathbb{K}$ is *correctible*.

Depending on the type of input and transmission model, the performance of a frame can be measured in deterministic or probabilistic ways. One measure is the worst-case for the reconstruction, which the maximal error norm among all reconstructed vectors. Since the error is proportional to the norm of the input vector, the operator norm $\|T^*(I - E_{\mathbb{K}})T\|$ can be chosen as a measure for the worst case error among all normalized inputs [5, 10, 2, 11]. Another possibility is a statistical performance measure such as the mean-squared error, where the average is either over unit-norm input vectors for specific erasures or over the combination of such input vectors and random erasures. We combine these performance measures in a unified notation, see e.g. [4].

8.1.2 Definition. Let $\mathbb{S}$ be the unit sphere in a real or complex N-dimensional Hilbert space $\mathcal{H}$, and let $\Omega = \{0, 1\}_{j=1}^{M}$ be the space of binary sequences of length M. Given a binary sequence $\omega = \{\omega_1, \omega_2, \dots, \omega_M\}$, we let the associated operator $E(\omega)$ be the diagonal $M \times M$ matrix with $E(\omega)_{j,j} = \omega_j$ for all $j \in \mathbb{J} = \{1, 2, \dots, M\}$. Let μ be a probability measure on the space $\mathbb{S} \times \Omega$, which is the product of the uniform probability measure on $\mathbb{S}$ and a probability measure on Ω. The p-th power error norm is given by

$$e_p(\Phi, \mu) = \left(\int_{\mathbb{S} \times \Omega} \|T^* E(\omega) T x - x\|^p d\mu(x, \omega) \right)^{1/p}$$

with the understanding that when $p = \infty$ it is the usual sup-norm. The quantity $e_\infty(\Phi, \mu)$ has also been called the worst-case error norm and $e_2(\Phi, \mu)^2$ is commonly referred to as the mean-squared error.

We conclude this section with remarks concerning the relationship between passive and active error correction for erasures when $p = \infty$. In principle, active error correction either results in perfect reconstruction or in an error that can only be controlled by the norm of the input, because $WE_{\mathbb{K}}T$ is an orthogonal projection if W is the Moore-Penrose pseudoinverse of $E_{\mathbb{K}}T$. This may make it seem as if the only relevant question for active error correction is whether $E_{\mathbb{K}}T$ has a left inverse.

However, even in cases where an erasure is correctible, numerical stability against round-off errors and other additive noise is desirable. This will be examined in more detail in Subsection **??**. We prepare the discussion there by a comparison of an error measure based on the Moore-Penrose pseudoinverse with the p-th power error norm. It turns out that if all erasures in Ω are correctible, then achieving optimality with respect to e_∞ is equivalent to minimizing the maximal operator norm among all Moore-Penrose pseudoinverses of $E(\omega)T, \omega \in \Omega$.

8.1.3 Definition. Let $\mathbb{J}$, $\mathbb{S}$ and Ω be as above, and let ν be the uniform probability measure on $\mathbb{S} \times \Omega$. Let $W(\omega)$ be the Moore-Penrose pseudoinverse of $E(\omega)T$, then we define

$$a_p(\Phi, \nu) = \left(\int_{\mathbb{S}\times\Omega} \|W(\omega)y\|^p d\nu(y, \omega) \right)^{1/p}$$

with the understanding that when $p = \infty$ it is the usual sup-norm.

8.1.4 Proposition. *Let $\mathcal{H}$ be an N-dimensional real or complex Hilbert space. For any set of erasures $\Gamma \subset \Omega$, let μ_Γ be the probability measure which is invariant with respect to the product of unitaries and permutations on $\mathbb{S} \times \Gamma$. Similarly, let ν be the probability measure on the unit sphere of $\ell^2(\{1, 2, \dots, M\}) \times \Gamma$, which is invariant with respect to the product of unitaries and permutations. If all erasures in Γ are correctible for a closed subset $\mathcal{S}$ of (M, N)-frames, then a frame Φ achieves the minimal worst-case error norm $e_\infty(\Phi, \mu) = \min_{\Psi \in \mathcal{S}} e_\infty(\Psi, \mu)$ if and only if it achieves the minimum $a_\infty(\Phi, \nu) = \min_{\Psi \in \mathcal{S}} a_\infty(\Psi, \nu)$.*

Proof. Let us fix an erasure E corresponding to a choice of $\omega \in \Gamma$. Given an isometry T (analysis operator of a Parseval frame), the left inverse to T with smallest operator norm is the (Hilbert) adjoint T^*. Given a Parseval frame and a diagonal projection E, then the operator norm of $T^*ET - I$ is the largest eigenvalue of $I - T^*ET$, because $T^*ET - I = T^*(E - I)T$ is negative definite. Factoring ET by polar decomposition into $VA = ET$, where A is non-negative and V is an isometry, so the operator norm of $A^{-1}T^*$ is $\|A^{-1}V^*VA^{-1}\|^{1/2} = \|A^{-2}\|^{1/2} = \|A^{-1}\|$, by positivity of A the inverse of the smallest eigenvalue of A, a_{min}.

This means, minimizing a_∞ with a fixed set of erasures amounts to maximizing the smallest eigenvalue appearing among the set of operators $\{A(\omega) : \omega \in \Gamma\}$. Comparing this with the error for blind reconstruction gives

$$\|(T^*ET - I)\| = \|I - A^*V^*VA\| = 1 - a_{min}^2.$$

Minimizing this error over Γ also amounts to maximizing the smallest eigenvalue. Thus, the minimization of e_∞ or a_∞ for a fixed set of erasures Γ is equivalent. $\square$

8.1.1 Hierarchical error models

Often it is assumed that losing one coefficient in the transmission process is rare, and that the occurence of two lost coefficients is much less likely. A similar hierarchy of probabilities usually holds for a higher number of losses. This motivates the design of frames following an inductive scheme: We require perfect reconstruction when no data is lost. Among the protocols giving perfect reconstruction, we want to minimize the maximal error in the case of one lost coefficient. Generally, we continue by choosing among the frames which are optimal for m erasures those performing best for $m+1$ erasures. For an alternative approach, which does not assume a hierarchy of errors, see the so-called maximally robust encoding [17] and the section on random Parseval frames further below.

8.1.5 Definition. Let $\mathcal{H}$ be an N-dimensional real or complex Hilbert space. We denote by $\mathcal{F}(M,N)$ the set of all (M,N)-frames, equipped with the natural topology from $\mathcal{H}^M$. Using as Γ_m the set of all m-erasures, $\Gamma_m = \{\omega \in \Omega : \sum_{j=1}^M \omega_j = m\}$, we let μ_m denote the product of uniform probability measures on $\mathbb{S} \times \Gamma_m$. We let $e_p^{(1)}(M,N) = \min\{e_p(\Phi,\mu_1) : \Phi \in \mathcal{F}(M,N)\}$ and $\mathcal{E}_p^{(1)}(M,N) = \{\Phi \in \mathcal{F}(M,N) : e_p(\Phi,\mu_m) = e_p^{(1)}(M,N)\}$. Proceeding inductively, we set for $1 \le m \le M$, $e_p^{(m)}(M,N) = \min\{e_m^p(\Phi,\mu_m) : \Phi \in \mathcal{E}_p^{(m-1)}(M,N)\}$ and and define the optimal *m-erasure frames for e_p* to be the nonempty compact subset $\mathcal{E}_p^{(m)}(M,N)$ of $\mathcal{E}_p^{(m-1)}(M,N)$ where the minimum of $e_p^{(m)}$ is attained.

In this manner, we obtain a decreasing family of frames which can be characterized in a geometric fashion. Results by Casazza and Kovačević [5] as reviewed in [10, Proposition 2.1] and slightly extended in [2] can be interpreted as the statement that among all Parseval frames, the equal-norm ones minimize the worst-case reconstruction error for one erasure.

8.1.6 Proposition. *For $1 < p \le \infty$, the set $\mathcal{E}_p^{(1)}(M,N)$ coincides with the family of equal-norm (M,N)-frames. Consequently, for $1 < p \le \infty, e_p^{(1)}(M,N) = N/M$.*

Proof. Given an (M,N)-frame $\Phi = \{\varphi_1,\dots,\varphi_M\}$ with analysis operator T, and a diagonal projection matrix D with one non-zero entry $D_{j,j}$, then $\|T^*DT\| = \|DTT^*D\| = \|\varphi_j\|^2$. If Φ is a Parseval frame, then $\sum_{j=1}^M \|f\|^2 = \operatorname{tr} TT^* = \operatorname{tr} T^*T = trI_N = N$, so minimizing the maximum norm among all frame vectors is achieved if and only if they all have the same norm. In this case, $\|\varphi_j\|^2 = N/M$, and thus $e_p^{(1)}(M,N) = N/M$ for any $p > 1$. □

Strohmer and Heath as well as Holmes and Paulsen [21, 10] showed that when they exist, equiangular Parseval frames are optimal for up to two erasures with respect to $e_\infty^{(2)}$. As stated by Holmes and Paulsen, if Φ is an equiangular (M,N)-frame, then TT^* is a self-adjoint rank N projection that can be written in the form $TT^* = aI + c_{M,N}Q$ where $a = N/M$, $c_{M,N} = \left(\frac{N(M-N)}{M^2(M-1)}\right)^{1/2}$, and the so-called signature matrix $Q = (Q_{i,j})$ is a self-adjoint $M \times M$ matrix satisfying $Q_{i,i} = 0$ for all i and for $i \ne j, |Q_{i,j}| = 1$. The proof of optimality uses that if D is a diagonal projection matrix with a 1 in the i-th and j-th diagonal entries and T is the analysis operator for an equal-norm (M,N)-frame $\Phi = \{\varphi_1,\dots,\varphi_M\}$, then $\|T^*DT\| = \|DTT^*D\| = N/M + |\langle\varphi_i,\varphi_j\rangle|$. Since $\sum_{j\ne l} |\langle\varphi_j,\varphi_l\rangle|^2 = \operatorname{tr}[(TT^*)^2] - \sum_{j=1}^M (TT^*)_{j,j}^2 = N - N^2/M$, the maximum magnitude among all the inner products cannot be below the average value, which gives a lower bound for the worst-case two-erasure. This bound is saturated if and only if all inner products have the same magnitude. Welch had established this inequality for unit-norm vector sequences [24].

The characterization of equiangular Parseval frames as optimal 2-erasure frames was extended to all sufficiently large values of p in [2].

8.1.7 Theorem ([2])**.** *If equiangular frames exist among the equal-norm (M,N)-frames and if $p > 2 + \left(\frac{5N(M-1)}{M-N}\right)^{1/2}$, then $\mathcal{E}_p^{(2)}(M,N)$ consists precisely of these equiangular frames.*

The existence of such equiangular Parseval frames for real Hilbert spaces depends on the existence of a matrix of $\pm 1's$ which satisfies certain algebraic equations. Thanks to the discovery by [21] of the connection between equiangular frames and the earlier work of Seidel and his collaborators in graph theory, much of the work on existence and construction of real equiangular tight frames benefits from known techniques. The construction of equiangular Parseval frames in the complex case was investigated with number-theoretic tools, see [25] and [11], as well as with a numerical scheme [22]. Recently, Seidel's combinatorial approach was extended to the complex case by considering signature matrices whose entries are roots of unity [3, 1, 9].

An averaging argument similar to the inequality by Welch was derived for the case of 3 erasures [**?**], in the context of fusion frames. We present the consequences for the special case of equiangular Parseval frames.

8.1.8 Theorem. *Let $M \geq 3$, $M > N$ and let Φ be an equiangular (M,N)-frame, then*

$$e_\infty^{(3)}(M,N) \geq \frac{N}{M} + 2c_{M,N}\cos(\theta/3)$$

where $\theta \in [-\pi,\pi]$ observes $\cos\theta = \frac{M-2N}{M(M-2)c_{M,N}}$. Equality holds if and only if $\mathrm{Re}[Q_{i,j}Q_{j,l}Q_{l,i}] = \cos(\theta)$ for all $i \neq j \neq l \neq i$, where Q is the signature matrix of Φ.

Proof. The operator norm of the submatrix of Q with rows and columns indexed by $\{i,j,l\}$ is $2\cos(\theta/3)$, with $\mathrm{Re}[Q_{i,j}Q_{j,l}Q_{l,i}] = \cos(\theta)$. However, the sum of all triple products is the constant

$$\sum_{i,j,l=1}^{M} Q_{i,j}Q_{j,l}Q_{l,i} = \frac{(M-1)(M-2N)}{c_{M,N}}$$

so the largest real part among all the triple products cannot be smaller than the average, which yields the desired inequality. □

8.1.9 Corollary. *Let M,N be such that $M > N$ and an equiangular (M,N)-frame exists with constant triple products $\mathrm{Re}[Q_{i,j}Q_{j,l}Q_{l,i}] = \cos(\theta)$ for some $\theta \in [-\pi,\pi]$, then the set $\mathcal{E}_\infty^{(3)}(M,N)$ contains precisely these frames.*

8.1.10 Remark. In [2], only $(M,M-1)$-frames were mentioned as examples for such 3-erasure optimal frames. However, recently Hoffman and Solazzo [9] found a family of examples which are not of this trivial type. We present one of their examples. It is the complex equiangular

$(8,4)$-frame with signature matrix

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -i & -i & -i & i & i & i \\ 1 & i & 0 & -i & i & -i & -i & i \\ 1 & i & i & 0 & -i & -i & i & -i \\ 1 & i & -i & i & 0 & i & -i & -i \\ 1 & -i & i & i & -i & 0 & -i & i \\ 1 & -i & i & -i & i & i & 0 & -i \\ 1 & -i & -i & i & i & -i & i & 0 \end{pmatrix}$$

In the real case Bodmann and Paulsen showed that, in fact, this 3-optimality condition is satisfied if and only if the frame is an equiangular $(M, M-1)$ or $(M, 1)$-frame. Thus, in order to differentiate between frames, they had to examine the case of equiangular tight frames in more detail. Bodmann and Paulsen [2] related the performance of these frames in the presence of higher numbers of erasures to graph theoretic quantities.

To this end, they established an upper bound for the error and characterized cases of equality in graph-theoretic terms: Let Φ be a real equiangular (M, N)-frame. Then $e_m^\infty(F) \leq N/M + (m-1)c_{M,N}$ with equality if and only if the signature matrix Q associated with Φ is the Seidel adjacency matrix of a graph that contains an induced complete bipartite subgraph on m vertices. The *Seidel adjacency matrix* of a graph G of M vertices is defined to be the $M \times M$ matrix $A = (a_{i,j})$ where $a_{i,j}$ is -1 when i and j are adjacent, it is $+1$ when i and j are not adjacent, and 0 when $i = j$. In certain cases Bodmann and Paulsen showed that as the size of a graph grows beyond some number, then among all the induced subgraphs of size up to 5 there is at least one complete bipartite graph. For such graphs, the worst-case m-erasure error is known up to $m = 5$. To differentiate between such types of equiangular Parseval frames, one needs to look beyond the 5-erasure error. Graph-theoretic criteria allowed the characterization of optimality by considering induced subgraphs of larger sizes [2].

8.1.2 Robustness of equiangular tight frames against erasures

We recall that when a frame with analysis operator T is used for the encoding, then an erasure is called correctible if ET has a left inverse, where E is the diagonal projection which sets the erased frame coefficients to zero. In this case, the left inverse effectively recovers any encoded vector x from the remaining set of non-zero frame coefficients.

The matrix ET has a left inverse if and only if all of its singular values are non-zero, or equivalently, whenever T^*ET is invertible. For Parseval frames this amounts to $\|I - T^*ET\| = \|T^*(I-E)T\| < 1$. This condition applies verbatim to sets of erasures, for example the set of diagonal projection matrices with m zeros on the diagonal representing erasures of m frame coefficients.

8.1.11 Definition. A Parseval frame Φ with analysis operator T is robust against m erasures if

$$\|T^*ET - I\| < 1$$

for each diagonal projection E with $\operatorname{tr} E = M - m$.

A sufficient criterion for robustness of real and complex equiangular Parseval frames uses the following error estimate, which is a special case of a result for fusion frames [**?**].

8.1.12 Theorem. *Let Φ be an equiangular (M,N)-frame with signature matrix Q, then*

$$e_\infty^{(m)}(M,N) \leq N/M + (m-1)c_{M,N}$$

with equality if and only if there exists a diagonal $M \times M$ unitary matrix Y such that Y^*QY contains an $m \times m$ principal submatrix with off-diagonal entries that are all 1's.

Proof. The largest eigenvalue of any $m \times m$ compression of Q determines the largest eigenvalue of the corresponding compression of TT^*. For each choice of m indices $\mathbb{K} = \{j_1, j_2, \dots, j_m\}$, a normalized eigenvector x of $(Q_{j,l})_{j,l\in\mathbb{K}}$ belonging to the largest eigenvalue maximizes $q(x) = \langle Qx, x\rangle$ among all unit vectors with support contained in $\mathbb{K}$. Using the Cauchy-Schwarz inequality gives $q(x) \leq \sum_{j\neq l} |x_j x_l| \leq (m-1)$, and equality occurs if and only if $Q_{j,l}x_l\overline{x_j} = 1/m$ for all $j \neq l$ in $\mathbb{K}$. Now we can pick Y such that $Y_{j,j} = x_j$ if $j \in \mathbb{K}$ and then Y^*QY is seen to have the claimed form. □

8.1.13 Corollary. *If $\frac{N}{M} + (m-1)c_{M,N} < 1$, then any equiangular (M,N)-frame is robust against m erasures.*

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