Functional Analysis, Math 7320
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1 Essentials of Topology

1.1 Continuity

Next we recall a stronger notion of continuity:

1.1.1 Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces, A map $f : X \to Y$ is called *uniformly continuous*, if

$$\forall \epsilon > 0, \exists \delta > 0, \text{ for all } p, q \in X, d_X(p, q) < \delta,$$

we have

$$d_Y(f(p), f(q)) < \epsilon.$$

1.1.2 Theorem. *Let X, Y be normed spaces, and d_X, d_Y are the metric induced by the norm of X, Y, If $A : X \to Y$ is linear continuous map, then A is uniformly continuous.*

Proof. Two steps to verify uniform continuity:

(1) Since A is continuous at 0, so

$$\forall \epsilon > 0 \exists \delta > 0, \text{s.th.} \forall p \in X, d_X(p, 0) < \delta,$$

we have

$$d_Y(A(p), 0) < \epsilon.$$

(2) We use this to prove uniform continuity: Let $\epsilon > 0$ be given and choose $\delta > 0$ as in (1), then

$$\forall p, q \in X, d_X(p, q) < \delta,$$

we have $z = p - q$ satisfies $d_X(z, 0) < \delta$, so $d_Y(A(z), 0) < \epsilon$. Now using the linearity of A gives

$$d_Y(A(p), A(q)) = \|A(p - q)\|_Y = d_Y(A(p - q), 0) < \epsilon$$

so A is uniformly continuous.

□

Next we make precise in which way vector space and norm structures are compatible:

1.1.3 Lemma. *Let X, Y be norm space, then $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ is a norm.*

Proof. We verify three properties of norm:

(1) Verify positive definiteness:

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y = 0,$$

implies by $\|x\|_X \geq 0$ and $\|y\|_Y \geq 0$ and the positive definiteness of the norms on X and Y

$$x = 0, y = 0.$$

so

$$(x,y) = 0.$$

(2) Verify scaling property from the homogeneity of the norms on X and Y:

$$\|\lambda(x,y)\|_{X\times Y} = \|\lambda x\|_X + \|\lambda y\|_Y = |\lambda|(\|x\|_X + \|y\|_Y) = |\lambda|\|(x,y)\|_{X\times Y}.$$

so

$$\|\lambda(x,y)\|_{X\times Y} = |\lambda|\|(x,y)\|_{X\times Y}.$$

(3) Verify triangle inequality from that of the norms on X and Y:

$$\|(x,y)+(p,q)\|_{X\times Y} = \|x+p\|_X+\|y+q\|_Y \leq \|x\|_X+\|p\|_X+\|y\|_Y+\|q\|_Y = \|(x,y)\|_{X\times Y}+\|(p,q)\|_{X\times Y}.$$

so

$$\|(x,y)+(p,q)\|_{X\times Y} \leq \|(x,y)\|_{X\times Y} + \|(p,q)\|_{X\times Y}.$$

this completes the proof.

□

1.1.4 Theorem. *Let $(X, \|.\|)$ be a norm space, then $f : X \times X \to X, (x,y) \to x+y$ is uniformly continuous. Scalar product: $\bullet : \mathbb{K} \times X \to X : (\lambda, x) \to \lambda x$ is continuous but not uniformly continuous.*

Proof. We prove this three statements:

(1) Since $\text{For } \epsilon > 0, \text{let } \delta = \epsilon > 0, \text{ then } \forall x, p \in X, y, q \in Y, \text{with } \|(x,y) - (p,q)\|_{X\times Y} = \|x-p\|_X + \|y-q\|_Y < \delta$, we have

$$\|(x+y) - (p+q)\|_{X\times Y} \leq \|x-p\|_X + \|y-q\|_Y < \delta = \epsilon.$$

so f is uniformly continuous.

(2) Since it is in metric space, so we only need to verify the sequence convergence:

let $\lambda_n, \lambda_0 \in \mathbb{K}, x_n, x_0 \in X$, with $x_n \to x_0, \lambda_n \to \lambda_0$.

by convergence of x_n, we have for

$$\epsilon = 1, \exists n_0 > 0 \text{ s.th. } \forall n > n_0, \|x_n - x_0\| \leq 1.$$

so

$$\forall n > n_0, \|x_n\| \leq \|x_n - x_0\| + \|x_0\| \leq 1 + \|x_0\|.$$

Hence, we can set $K = \max\{\|x_n\|, \|x_0\| + 1, n \leq n_0\}$, then by convergence of λ_n, x_n, we have

$$\forall \epsilon, \exists n_1 \in \mathbb{N} \text{ s.th. } \forall n > n_1, |\lambda_n - \lambda_0| \leq \epsilon/2K.$$

and

$$\forall \epsilon, \exists n_2 > n_1, \forall n > n_2, \|x_n - x_0\| \leq \epsilon/2(|\lambda_0| + 1).$$

so

$$\forall \epsilon, \exists n_2, \forall n > n_2,$$
$$|\lambda_n x_n - \lambda_0 x_0| \leq |(\lambda_n - \lambda_0)|\|x_n\| + |\lambda_0|\|(x_n - x_0)\| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

this completes the proof.

(3) Scalar product is not uniform. Assume this were the case, then for a given $\epsilon > 0$, there would be $\delta > 0$ such that any pair of points at distance at most δ would be mapped to a pair at a distance at most ϵ. Take x_0 a non zero element, and let $\epsilon = 2\|x_0\|$. Then take $\lambda_{1,n} = n, \lambda_{2,n} = n + 1/n, x_{1,n} = nx_0, x_{2,n} = nx_0$, then

$$\|(\lambda_{1,n}, x_{1,n}) - (\lambda_{2,n}, x_{2,n}))\| = 1/n \to 0,$$

which becomes smaller than any $\delta > 0$ but for any n

$$\lambda_{1,n} x_{1,n} - \lambda_{2,n} x_{2,n} = -x_0$$

and hence $\|\lambda_{1,n} x_{1,n} - \lambda_{2,n} x_{2,n}\| = \|x_0\| > \epsilon$. By contradiction, the scalar product is not uniformly continuous.

□

1.2 Completeness

Next we talk about completeness:

1.2.5 Definition. Let (X, τ) be topological space, $(x_n)_{n \in \mathbb{N}} \subseteq X$, we say $x_n \to x$ in X, if for each $U \in \mathcal{U}_x$(neighourhood of x), $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geqslant n_0, x_n \in U$.

1.2.6 Remark. For metric spaces, this implies the usual form of convergence:

$$\forall \epsilon, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, d(x_n, x) < \epsilon.$$

For metric space, we will have a weaker notion of 'convergence' called Cauchy property.

1.2.7 Definition. Let (X, d) be metric space, a sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence if:

$$\forall\epsilon, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, d(x_n, x_m) \leq \epsilon.$$

1.2.8 Remark. A convergence sequence is Cauchy by triangle inequality:
if

$$\forall\epsilon, \exists n_0, \forall n > n_0, d(x_n, x) < \epsilon/2.$$

then

$$\forall\epsilon, \exists n_0, \forall m, n > n_0, d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

but converse is not true for some space$X$ like open interval equipped with usual metric.

1.2.9 Definition. (X, d) be metric space is called complete if each Cauchy sequence converges in X. A complete normed space is called Banach Space.

1.2.10 Remark. A subset of a complete metric space is completeness iff it is closed(sees next lemma for proof).

1.2.11 Lemma. *Let (X, d) be complete metric space and $Y \subseteq X$, then (Y, d) is complete iff Y is closed in X*

Proof. We prove it by two steps:

(1) if Y is a closed in X, cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset Y$.

so

$$(x_n)_{n\in\mathbb{N}} \subset X.$$

Since X is complete,

$$\exists x \in X, x_n \to x.$$

Since Y is closed, so the limit

$$x \in Y.$$

so Y is complete.

(2) Assume Y is complete and take any $x \in \overline{Y}$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in Y such that

$$x_n \to x \in X$$

Since this sequence converges in X, then

$$\forall\epsilon, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, d(x_n, x) \leq \epsilon/2.$$

then

$$\forall\epsilon, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, d(x_n, x_m) \leq d(x_m, x) + d(x_n, x) \leq \epsilon.$$

so this sequence is also cauchy in Y.

So it is Cauchy in Y and by the completeness of Y, it converges in Y to the limit $x \in Y$. Thus the limit $x \in Y$, so Y is closed.

□

1.2.12 Example. We give some concrete examples:

(a) $\mathbb{K} = \mathbb{C}$or $\mathbb{R}$ are complete metric space.

(b) $\mathbb{Q} \subset \mathbb{R}$ is not complete.

(c) $(\mathbb{K}^n, \|.\|_p)$, $1 \leq p \leq \infty$ is Banach space.

(d) l^p, $1 \leq p \leq \infty$ with norm $\|x\|_p = (\sum |x_j|^p)^{1/p}$ is a Banach Space as well as $c_0 \subset l^\infty$

(e) Bounded function space $\mathbb{B}(X, \mathbb{K})$ is a Banach Space equipped with $\|.\|_\infty$.

(f) if X is a norm space, Y is Banach Space, then $\mathbb{B}(X, Y)$ is Banach Space with the operator norm.

Proof. Now we prove all of them:

(a) For number field $\mathbb{K}$ like $\mathbb{C}, \mathbb{R}$, since $\mathbb{C} \cong \mathbb{R}^2$, so it deduced to the case of $\mathbb{R}$, but I do not how to prove $\mathbb{R}$ is complete because $\mathbb{R}$ is originally defined as the equivalence class of cauchy sequence in $\mathbb{Q}$, that is, $\mathbb{R}$ is defined as the completeness of $\mathbb{Q}$.

(b) We prove that $\mathbb{Q}$ is not closed in $\mathbb{R}$. Every irrational number can be expressed as binary form:

$$x = \sum_{j=1}^{\infty} a_i/2^i, a_i = 0/1.$$

so consider the finite term:

$$x_n = \sum_{j=1}^{n} a_i/2^i, a_i = 0/1.$$

we note that $x_n \to x$ and $x_n \in \mathbb{Q}$, which complete the proof.

(c) This is special case of (d), so we turn to (d) first. *(How is (c) embedded in (d)?) You need to use that it is a closed subset to relegate it to (d).*

(d) For $p < \infty$, first we prove it is a norm space.

(1) if $\|x\|_p = (\sum_{j=1}^{\infty} |x_{n,j})^p|^{1/p} = 0$, then $\forall n, x_n = 0$, so $x = 0$.

(2) $\|\lambda x\|_p = (\sum_{j=1}^{\infty} |\lambda x_{n,j})^p|^{1/p} = |\lambda| \|x\|_p$.

(3) by Minkowskii inequality,

$$\|x + y\|_p = (\sum_{j=1}^{\infty} |x_{n,j} + y_{n,j}|^p)^{1/p} \leq (\sum_{j=1}^{\infty} |x_{n,j}|^p)^{1/p} + (\sum_{j=1}^{\infty} |y_{n,j}|^p)^{1/p} = \|x\|_p + \|y\|_p.$$

this complete the proof of norm properties.

We turn to completeness:

let $(x_n)_{n\in\mathbb{N}} \in l^p$ is a cauchy sequence, that is:

$$\forall\epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, (\sum_{j=1}^{\infty} |x_{n,j} - x_{m,j}|^p)^{1/p} < \epsilon/2.$$

so for each j, we have

$$\forall\epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, (|x_{n,j} - x_{m,j}|^p)^{1/p} < \epsilon/2.$$

that is, so for each j, $(x_{n,j})_{n\in\mathbb{N}}$ is cauchy sequence. so

$$\exists y_j, x_{n,j} \to y_j.$$

For fixed $n \geq n_0$, we apply Fatou lemma to

$$(\sum_{j=1}^{\infty} |x_{n,j} - x_{m,j}|^p)^{1/p} < \epsilon.$$

when $m \to \infty$,

$$(\sum_{j=1}^{\infty} |x_{n,j} - y_j|^p)^{1/p} = (\sum_{j=1}^{\infty} \varliminf_{m\to\infty} |x_{n,j} - x_{m,j})^p|^{1/p} \leqslant \varliminf_{m\to\infty} (\sum_{j=1}^{\infty} |x_{n,j} - x_{m,j}|^p)^{1/p} \leq \epsilon/2.$$

that is

$$\forall\epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, (\sum_j |x_{n,j} - y_j|^p)^{1/p} \leq \epsilon/2 < \epsilon.$$

Moreover, by Minkowski's inequality:

$$(\sum_j |y_j|^p)^{1/p} \leq (\sum_j |x_{n_0,j} - y_j|^p)^{1/p} + (\sum_j |x_{n_0,j}|^p)^{1/p} < \epsilon + (\sum_j |x_{n_0,j}|^p)^{1/p} < \infty.$$

so $y \in \ell^p$.

Thus, we have shown in l^p,

$$x_n \to y.$$

This complete the proof of completeness for case $p < \infty$.

A similar proof can be applied to the case when $p = \infty$:

For $p = \infty$, first we prove it is a norm space.

(1) if $\|x\|_\infty = \sup_n |x_n| = 0$, then $\forall n, x_n = 0$, so $x = 0$.

(2) $\|\lambda x\|_\infty = \sup_n |\lambda x_n| = |\lambda| \|x\|_\infty$.

(3) triangle inequality:

$$\|x+y\|_\infty = \sup_n |x_n + y_n| \le \sup_n |x_n| + \sup_n |y_n| = \|x\|_\infty + \|y\|_\infty$$

this complete the proof of norm properties.

We turn to completeness:

let $(x_n)_{n\in\mathbb{N}} \in l^\infty$ is a cauchy sequence, that is:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, \sup_j |x_{n,j} - x_{m,j}| < \epsilon/2.$$

so for each j, we have

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, |x_{n,j} - x_{m,j}| < \epsilon/2.$$

that is, so for each j, $(x_{n,j})_{n\in\mathbb{N}}$ is cauchy sequence. so

$$\exists y_j, x_{n,j} \to y_j.$$

For fixed $n \ge n_0$, we let $m \to \infty$,

$$\forall \epsilon, \exists n_0, \forall n > n_0, \forall j, |x_{n,j} - y_j| = \lim_{m\to\infty} |x_{n,j} - x_{m,j}| \le \epsilon/2 < \epsilon.$$

that is

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, \sup_j |x_{n,j} - y_j| \le \epsilon/2 < \epsilon.$$

Moreover, by triangular inequality:

$$\sup_j |y_j| \le \sup_j |x_{n_0,j} - y_j| + \sup_j |x_{n_0,j}| < \epsilon + \sup_j |x_{n_0,j}| < \infty.$$

so $y \in \ell^\infty$.

Thus, we have shown in l^∞,

$$x_n \to y.$$

This complete the proof of completeness for case $p = \infty$.

so l^∞ is also complete.

c_0 is closed with respect to the norm in l^∞:

Assume a sequence in c_0 and a limit in l^∞

$$x_n \to x.$$

that is

$$\forall \epsilon, \exists n_0, \forall n > n_0, \forall j, |x_{n,j} - x_j| < \epsilon/2.$$

Fix $n \ge n_0$ then letting $j \to \infty$, we have by $\lim_{j\to\infty} x_{n,j} = 0$ that

$$\overline{\lim_{j\to\infty}} |x_j| = \overline{\lim_{j\to\infty}} |x_{n,j} - x_j| \le \epsilon/2 < \epsilon.$$

so $x \in c_0$, that is c_0 is a closed space in l^∞. Consequently, c_0 is also Banach Space. Returning to (c), consider $\mathbb{K}^n \subset l^p$ i.e. $\mathbb{K}^n = \{x \in l^p : x_j = 0, j \geq n+1\}$. assume in l^p, we have $x_n \in \mathbb{K}^n \to x$, that is:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, (\sum_j |x_{n,j} - x_j|^p)^{1/p} < \epsilon/2.$$

then we have:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \geqslant n_0, (\sum_{j \geq n+1} |x_j|^p)^{1/p} = (\sum_{j \geq n+1} |x_{n,j} - x_j|^p)^{1/p} < \epsilon/2.$$

that means:

$$x_j = 0, j \geq n+1.$$

so $x \in \mathbb{K}^n$ i.e. $\mathbb{K}^n$ is closed subspace in l^p, so $\mathbb{K}^n$ is complete.

$\square$