The following items are the continuation of a list of examples of the previous set of notes:

(e') If $X$ is a metric space, then $(C_b(X, K), \| \cdot \|_\infty)$ is a Banach space, where $C_b(X, K)$ is the set of continuous, bounded, $K$-valued functions on $X$ and $\| \cdot \|_\infty$ is defined by $f \mapsto \sup_{x \in X} |f(x)|$.

To see that $\| \cdot \|_\infty$ is a norm, observe that

$$\|f\|_\infty = \sup_{x \in X} |f(x)| = 0 \iff f(x) = 0 \text{ for all } x \in X.$$  

Moreover, if $\lambda \in \mathbb{R}$, then $\|\lambda f\|_\infty = \sup_{x \in X} |\lambda f(x)| = |\lambda| \sup_{x \in X} |f(x)| = |\lambda| \|f\|_\infty$.

Finally, if $f, g \in C_b(X, \mathbb{R})$, then

$$\|f + g\|_\infty = \sup_{x \in X} |f(x) + g(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| = \|f\|_\infty + \|g\|_\infty.$$  

To see that $(C_b(X, \mathbb{R}), \| \cdot \|_\infty)$ is complete, let $(f_n)_{n \in \mathbb{N}}$ be Cauchy in $C_b(X, \mathbb{R})$ and let $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$ whenever $m, n \geq n_0$, which implies that for all $x \in X$, $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \varepsilon$ whenever $m, n \geq n_0$, which in turn implies that $(f_n(x))_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ and thus converges to an element of $\mathbb{R}$. Define $f : X \to \mathbb{R}$ by $x \mapsto \lim_{n \to \infty} f_n(x)$.

To see that $(f_n)_{n \in \mathbb{N}}$ converges to $f$ and that $f$ is continuous, let $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $\|f_m - f_n\|_\infty < \varepsilon/2$ whenever $m, n \geq n_0$, which implies that for all $x \in X$, $|f_m(x) - f_n(x)| < \varepsilon/2$ whenever $m, n \geq n_0$, which in turn implies that for all $x \in X$,

$$\lim_{m \to \infty} |f_m(x) - f_n(x)| = |f(x) - f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon$$

whenever $n \geq n_0$. Therefore, $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $f$, which implies that $f$ is continuous.

To see that $f$ is bounded, let $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $\|f - f_{n_0}\| < \varepsilon$, and note that $f_{n_0}$ is bounded, which implies that there is an $M \geq 0$ such that $\|f_{n_0}\|_\infty \leq M$. Then

$$\|f\|_\infty \leq \|f - f_{n_0}\|_\infty + \|f_{n_0}\|_\infty < \varepsilon + M.$$  

Therefore, $f$ is bounded.
(f) If \((X, \| \cdot \|_X)\) is a normed space and \((Y, \| \cdot \|_Y)\) is a Banach space, then \((B(X, Y), \| \cdot \|_\text{op})\) is a Banach space, where \(\| \cdot \|_\text{op}\) is the operator norm defined by

\[ S \mapsto \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X}. \]

To see that \(B(X, Y)\) is a normed space, let \(S, T \in B(X, Y)\). Then

\[ \|S + T\|_\text{op} = \sup_{x \neq 0} \frac{\|(S + T)x\|_Y}{\|x\|_X} \leq \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} + \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \|S\|_\text{op} + \|T\|_\text{op}. \]

Let \(\lambda \in \mathbb{K}\). Then

\[ \|\lambda S\|_\text{op} = \sup_{x \neq 0} \frac{\|\lambda Sx\|_Y}{\|x\|_X} = |\lambda| \sup_{x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} = |\lambda| \|S\|_\text{op}. \]

Observe that

\[ 0 = \|T\|_\text{op} = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \iff \|Tx\|_Y = 0 \text{ for all } 0 \neq x \in X \iff T = 0. \]

To see that \(B(X, Y)\) is complete, let \((T_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \(B(X, Y)\), let \(\varepsilon > 0\), and let \(x \in X\). Then there is an \(n_0 \in \mathbb{N}\) such that \(\|T_m - T_n\|_\text{op} < \varepsilon / \|x\|_X\) whenever \(m, n \geq n_0\), which implies that \(\|T_m x - T_n x\|_Y \leq \|T_m - T_n\|_\text{op} \|x\|_X < \varepsilon\), which in turn implies that \((T_n x)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(Y\) and thus converges to an element in \(Y\) since \(Y\) is complete. Therefore, define \(T : X \to Y\) by \(x \mapsto \lim_{n \to \infty} T_n x\).

To see that \(T\) is linear, let \(x, y \in X\) and let \(\lambda \in \mathbb{K}\). Then

\[ T(\lambda x + y) = \lim_{n \to \infty} T_n (\lambda x + y) = \lambda \lim_{n \to \infty} T_n x + \lim_{n \to \infty} T_n y = \lambda T x + T y. \]

To see that \(T\) is bounded, let \(\varepsilon = 1\) and let \(x \in X\) such that \(\|x\|_X \leq 1\). Then there is an \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\),

\[ \|T x\|_Y \leq \|T_n x\|_Y + \|T x - T_n x\|_Y < \|T_n\|_\text{op} + 1. \]

To see that \(\lim_{n \to \infty} T_n = T\), let \(\varepsilon > 0\) and let \(x \in X\) such that \(\|x\|_X \leq 1\). Then there is an \(n_0 \in \mathbb{N}\) such that \(\|T x - T_n x\|_Y < \varepsilon / 2\) whenever \(n \geq n_0\). Since \((T_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(B(X, Y)\), there is an \(N_0 \in \mathbb{N}\) such that \(\|T_m - T_n\|_\text{op} < \varepsilon / 2\) whenever \(m, n \geq N_0\). Therefore,

\[ \|T x - T_n x\|_Y \leq \|T x - T_m x\|_Y + \|T_m x - T_n x\|_Y < \varepsilon \]

whenever \(m, n \geq \max\{n_0, N_0\}\).
1.3 Completion

1.3.1 Definition. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(f : X \to Y\).

1. If \(d_Y(f(x), f(y)) = d_X(x, y)\) for all \(x, y \in X\), then \(f\) is called an \textbf{isometry}.

2. If \(f(X) = Y\) and \(f\) is an isometry, then \(f\) is called an \textbf{isometric isomorphism}, and we write \(X \cong Y\).

3. If \(f\) is an isometry, \(\overline{f(X)} = Y\), and \(Y\) is complete, then \(f\) is called a \textbf{completion}.

1.3.2 Theorem. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces.

1. \(X\) has a completion.

2. If \(f : X \to Y\) is uniformly continuous, then there is a unique uniformly continuous map \(\hat{f} : \hat{X} \to \hat{Y}\) such that \(\hat{f}|_X = f\), where \(\eta : X \to (\hat{X}, d_{\hat{X}})\) and \(\varphi : Y \to (\hat{Y}, d_{\hat{Y}})\) are completions.

3. If \(X\) has completions \(\hat{X}\) and \(\eta : X \to Y\), then \(\hat{X}\) and \(Y\) are isometrically isomorphic.

Proof. (1) Fix \(x_0 \in X\) and define \(\eta : X \to C_b(X, \mathbb{R})\) by \(x \mapsto f_x\), where \(f_x : X \to \mathbb{R}\) is defined by \(y \mapsto d_X(x, y) - d_X(x_0, x)\). Recall that \(C_b(X, \mathbb{R})\) is a metric space with the metric induced by \(\| \cdot \|_\infty\). Since \(f_x(y) = d_X(x, y) - d_X(x, x_0) \leq d_X(x, x_0)\), \(f_x\) is bounded. Since \(d_X\) is continuous, \(f_x\) is continuous. Therefore, \(f_x \in C_b(X, \mathbb{R})\). Let \(x_1, x_2, y \in X\). Then \(f_{x_1}(y) - f_{x_2}(y) = d_X(x_1, y) - d_X(x_2, y)\), which implies that \(|f_{x_1}(y) - f_{x_2}(y)| \leq d_X(x_1, x_2)\). Observe that equality is attained if \(y = x_2\). Therefore, \(\|f_{x_1} - f_{x_2}\|_\infty = d_X(x_1, x_2)\), which implies that \(\eta\) is an isometry. Let \(\hat{X} = \eta(X)\). Since \((C_b(X, \mathbb{R}), \| \cdot \|_\infty)\) is a Banach space and \(\hat{X}\) is closed, \(\hat{X}\) is complete, which implies that \(\hat{X}\) is a completion of \(X\).

(2) Identify \(X\) with \(\eta(X)\), identify \(Y\) with \(\varphi(Y)\), let \(\hat{x} \in \hat{X}\), and let \(\varepsilon > 0\). Since \(X\) is dense in \(\hat{X}\), there is a sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) that converges to \(\hat{x}\). Since \(f\) is uniformly continuous, there is a \(\delta > 0\) such that for all \(x, x' \in X\) with \(d_X(x, x') < \delta\), \(d_Y(f(x), f(x')) < \varepsilon\). Since \((x_n)_{n \in \mathbb{N}}\) is Cauchy, there is an \(n_0 \in \mathbb{N}\) such that \(d_X(x_m, x_n) < \delta\) whenever \(m, n \geq n_0\), which implies that \(d_{\hat{Y}}(\hat{f}(x_m), \hat{f}(x_n)) < \varepsilon\) whenever \(m, n \geq n_0\), which in turn implies that \((\hat{f}(x_n))_{n \in \mathbb{N}}\) is Cauchy and thus converges to an element of \(\hat{Y}\). Extend \(f\) to \(\hat{f}\) by defining \(\hat{f}(\hat{x}) = \lim_{n \to \infty} \hat{f}(x_n)\).

To see that \(\hat{f}\) is well defined, let \((x'_n)_{n \in \mathbb{N}}\) be a sequence in \(X\) that converges to \(\hat{x}\). Then, by the previous argument, \((\hat{f}(x'_n))_{n \in \mathbb{N}}\) is Cauchy, which implies that \((\hat{f}(x_1), \hat{f}(x'_1), \hat{f}(x_2), \hat{f}(x'_2), \ldots)\) is Cauchy and converges to \(\hat{x}\) since its subsequence \((\hat{f}(x_m))_{n \in \mathbb{N}}\) converges to \(\hat{x}\), which in turn implies that \((\hat{f}(x'_n))_{n \in \mathbb{N}}\) converges to \(\hat{x}\).

To see that \(\hat{f}|_X = f\), let \(x \in X\) and let \((x_n)_{n \in \mathbb{N}}\) be such that \(x_n = x\) for all \(n \in \mathbb{N}\). Then \(\hat{f}(x) = \lim_{n \to \infty} \hat{f}(x_n) = f(x)\).

To see that \(\hat{f}\) is uniformly continuous, let \(\varepsilon > 0\). Since \(\hat{f}|_X\) is uniformly continuous, there is a \(\delta > 0\) such that for all \(x, y \in X\) with \(d_X(x, y) < \delta\), \(d_Y(\hat{f}(x), \hat{f}(y)) < \varepsilon/3\). Let \(\hat{x}, \hat{y} \in \hat{X}\) with \(d_{\hat{X}}(\hat{x}, \hat{y}) < \delta/3\). Then there are sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) in \(X\) converging to \(\hat{x}\)
and \( \hat{y} \), respectively, which implies that there is an \( n_0 \in \mathbb{N} \) such that \( d_X(x_n, \hat{x}) < \delta/3 \) and \( d_X(y_n, \hat{y}) < \delta/3 \) whenever \( n \geq n_0 \). Since \( d_X(x_n, y_n) \leq d_X(x_n, \hat{x}) + d_X(\hat{x}, y_n) < \delta \) whenever \( n \geq n_0 \), \( d_Y(\hat{f}(x_n), \hat{f}(y_n)) < \varepsilon/3 \) whenever \( n \geq n_0 \). Since \( (\hat{f}(x_n))_{n \in \mathbb{N}} \) and \( (\hat{f}(y_n))_{n \in \mathbb{N}} \) converge to \( \hat{f}(\hat{x}) \) and \( \hat{f}(\hat{y}) \), respectively, there is an \( N_0 \in \mathbb{N} \) such that \( d_Y(\hat{f}(x_n), \hat{f}(\hat{x})) < \varepsilon/3 \) and \( d_Y(\hat{f}(y_n), \hat{f}(\hat{y})) < \varepsilon/3 \) whenever \( n \geq N_0 \). Letting \( N = \max\{n_0, N_0\} \) yields that \( d_Y(\hat{f}(\hat{x}), \hat{f}(\hat{y})) \leq d_Y(\hat{f}(\hat{x}), \hat{f}(x_n)) + d_Y(\hat{f}(x_n), \hat{f}(y_n)) + d_Y(\hat{f}(y_n), \hat{f}(\hat{y})) < \varepsilon \).

To see that \( \hat{f} \) is unique, let \( h : \hat{X} \rightarrow \hat{Y} \) be a uniformly continuous extension of \( f \) and let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( X \) converging to \( \hat{x} \in \hat{X} \). Since \( h \) is continuous, \( \lim_{n \to \infty} h(x_n) = h(\lim_{n \to \infty} x_n) = h(\hat{x}) \). Since \( h(x_n) = f(x_n) \) for all \( n \in \mathbb{N} \), \( h(\hat{x}) = \lim_{n \to \infty} h(x_n) = \lim_{n \to \infty} f(x_n) = \hat{f}(\hat{x}) \).

(3) Identify \( X \) with \( \eta(X) \). To see that \( \eta \) is uniformly continuous, let \( \varepsilon = \delta > 0 \) and observe that if \( x, y \in X \) are such that \( d_X(x, y) < \delta \), then \( d_Y(\eta(x), \eta(y)) = d_X(x, y) < \delta = \varepsilon \) since \( \eta \) is an isometry. Moreover, observe that the completion of a complete metric space is its identity map. Therefore, \( \eta \) extends to \( \hat{\eta} : \hat{X} \rightarrow Y \).

To see that \( \hat{\eta} \) is an isometry, let \( \hat{x}, \hat{y} \in \hat{X} \). Then there are sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) in \( X \) that converge to \( \hat{x} \) and \( \hat{y} \), respectively. Therefore,

\[
d_X(x_n, y_n) = d_Y(\eta(x_n), \eta(y_n)) = d_Y(\eta(x_n), \eta(y_n)),
\]

and taking the limit as \( n \) approaches infinity yields that \( d_X(\hat{x}, \hat{y}) = d_Y(\eta(\hat{x}), \eta(\hat{y})) \).

As above, \( \eta^{-1} : \eta(X) \rightarrow X \) extends to an isometry \( \hat{\eta}^{-1} : Y \rightarrow \hat{X} \). Therefore, \( \hat{\eta}^{-1} \circ \hat{\eta} : \hat{X} \rightarrow \hat{X} \) is an extension of the identity map on \( X \), which implies that \( \hat{\eta}^{-1} \circ \hat{\eta} : \hat{X} \rightarrow \hat{X} \) is the identity map on \( \hat{X} \). Similarly, \( \hat{\eta} \circ \hat{\eta}^{-1} : Y \rightarrow Y \) is the identity map on \( Y \). Therefore, \( \hat{\eta} \) is an isometric isomorphism.

To visualize the functions \( f_x \) constructed in the proof of (1) above, let \( X = \mathbb{Q} \) and let \( x_0 = 0 \). Then the graphs of \( f_3 \) and \( f_2 \) are as follows:

![Plots of f3 and f2 with x0=0.](image-url)
Moreover, the graph of \(|f_3 - f_2|\) is as follows:

\[
\text{Plot of } |f_3 - f_2| \text{ with } x_0 = 0.
\]

Therefore, \(\sup_{x \in \mathbb{Q}} |f_3(x) - f_2(x)| = 5\), which shows us that \(\quad | - 3 - 2| = \|f_3 - f_2\|_{\infty}\).