1.6.52 Lemma. Let \((X, \tau)\) be a topological space and let \(Y \subseteq X\).

(a) \(Y\) is quasi-compact with respect to the trace topology if and only if each open cover of \(Y\) in \(X\) has a finite subcover.

(b1) If \(X\) is Hausdorff and \(Y\) is compact, then \(Y\) is closed.

(b2) If \(X\) is compact and \(Y\) is closed, then \(Y\) is compact.

(c) If \(X\) is normed and \(Y\) is compact, then \(Y\) is bounded.

Proof. (a) Proven in the previous lecture.

(b1) We will show the complement \(X \setminus Y\) is open in \(X\). To this end, let \(x \in X \setminus Y\). Now for each \(y \in Y\), since \(X\) is Hausdorff, there are disjoint open sets \(U_y\) and \(V_y\) containing \(x\) and \(y\), respectively. We then have \(Y \subseteq \bigcup_{y \in Y} V_y\), and since \(\{V_y\}_{y \in Y}\) is an open cover and \(Y\) is compact, there is a finite subcover \((V_y)_{y \in F}\) of \(Y\) with \(|F| < \infty\) and

\[
\bigcup_{y \in F} V_y \cap \left(\bigcap_{y \in F} U_y\right) = \emptyset,
\]

since every \(V_y\) is disjoint from \(U_y\). But as a finite intersection of open sets, \(\bigcap_{y \in F} U_y\) is open and contains \(x\). So \(\bigcap_{y \in F} U_y \subseteq X \setminus Y\), which implies \(X \setminus Y\) is open in \(X\).

(b2) Let \(X\) be compact and let \(Y\) be closed. Then \(X \setminus Y\) is open in \(X\) and given an open cover \((U_i)_{i \in I}\) of \(Y\), composed by open sets in \((X, \tau)\), we see that \((\bigcup_{i \in I} U_i) \cup (X \setminus Y)\) is an open cover of \(X\). Now since \(X\) is compact, there is a finite subcover of \(X\) which also covers \(Y\). If \(X \setminus Y\) is an element of this finite subcover, then by removing it, we have found a finite subcover of \(Y\). We infer from this property that \((Y, \tau_Y)\) has the Heine-Borel/finite subcover property as well, because every set \(V \subseteq Y\) that is open in \(\tau_Y\) is obtained from \(V = Y \cap U\) with \(U\) open in \(X\). Thus, after “lifting” the open cover with respect to \(\tau_Y\) to an open cover in \(X\) and passing to a finite subcover, intersecting the sets with \(Y\) again gives the desired finite open subcover in \((Y, \tau_Y)\).

It remains to show the Hausdorff property to establish compactness. Given \(x, y \in Y\) with \(x \neq y\), there are open (in \(X\)) sets \(V_x\) and \(V_y\) with \(x \in V_x, y \in V_y\) and \(V_x \cap V_y = \emptyset\). Thus, \(V_x \cap Y\) and \(V_y \cap Y\) have the desired separation properties to make \((Y, \tau_Y)\) Hausdorff. Hence \((Y, \tau_Y)\) is compact.
In a normed space, if \((B_n(0))_{n \in \mathbb{N}}\) is an open cover of \(X\), then it also covers \(Y\). Thus there is a finite subcover \((B_n(0))_{n \in F}\) of \(Y\) with \(|F| < \infty\), and by choosing \(N = \max\{n \in F\}\) we have
\[
Y \subset B_N(0) \subset \overline{B}_N(0),
\]
which means \(Y\) is bounded. \(\square\)

2 Topological Vector Spaces

2.1 Fundamental properties

2.1.1 Definition. A vector space \(X\) together with a topology \(\tau\) is called a topological vector space if

1. for every point \(x \in X\), the singleton \(\{x\}\) is a closed set.
2. the vector space operations
\[
+: X \times X \to X, \quad (x, y) \mapsto x + y
\]
and
\[
\cdot : \mathbb{K} \times X \to X, \quad (\lambda, x) \mapsto \lambda x
\]
are continuous with respect to the product topology on \(X \times X\) and \(\mathbb{K} \times X\), respectively.

2.1.2 Theorem. Every normed space is a topological vector space.

Proof. As shown before, '+' and '.' are continuous operations. Moreover,
\[
\bigcap_{n \in \mathbb{N}} B_{1/n}(x) = \{x\}
\]
is closed as an intersection of closed sets. \(\square\)

2.1.3 Remark. (a) For \(a, b \in X\), let \(V_a \in \mathcal{U}(a)\) and \(V_b \in \mathcal{U}(b)\) be open sets. Since each neighborhood of \((a, b) \in X \times X\) contains \(V_a \times V_b\), continuity of '+' means that for each \(U \in \mathcal{U}(a + b)\), we can find \(V_a, V_b\) as above with
\[
V_a + V_b = \{a' + b' : a' \in V_a, b' \in V_b\} \subset U.
\]

(b) Analogously, since \(\mathbb{K}\) is equipped with the topology of open balls, for \(U \in \mathcal{U}(\lambda x)\), there is an open \(V_x \in \mathcal{U}(x)\) and \(\delta > 0\) such that
\[
B_\delta(\lambda)V_x = \{\lambda'x' : \lambda' \in B_\delta(\lambda), x' \in V_x\} \subset U.
\]

Next, we explore implications of continuity for the topological structure of the space.
2.1.4 Theorem. Let \( X \) be a topological vector space, \( a \in X \) and \( \lambda \in \mathbb{K} \) with \( \lambda \neq 0 \). Then both the translation operator \( T_a : X \to X \) with \( T_ax = x + a \) and the scaling operator \( M_\lambda : X \to X \) with \( M_\lambda x = \lambda x \) are homeomorphisms of \( X \) onto \( X \).

Proof. For \( a \in X \) and \( 0 \neq \lambda \in \mathbb{K} \), we note that \( T_{-a} \circ T_a = \text{id} \) and that \( M_{\lambda^{-1}} \circ M_\lambda = \text{id} \), so \( T_a \) and \( M_\lambda \) are continuous. Hence, it is sufficient to show that for each \( a \in X \) and \( 0 \neq \lambda \in \mathbb{K} \), \( T_a \) and \( M_\lambda \) are continuous. By the continuity of `+', given \( x, a \in X \), then for \( U \in \mathcal{U}(x + a) \) there are \( V_a \in \mathcal{U}(a), V_x \in \mathcal{U}(x) \) such that \( V_a + V_x \subseteq U \) and hence \( a + V_x \subseteq U \). This means \( T_a(V_x) \subseteq U \) and so \( T_a \) is continuous at \( x \) and since \( x \) was arbitrary, \( T_a \) is continuous.

Similarly, given \( U \in \mathcal{U}(\lambda x) \), there is \( V_x \) and \( \delta > 0 \) such that \( B_\delta(\lambda)V_x \subseteq U \), which means \( \lambda V_x \subseteq U \), and thus for each \( \lambda \neq 0 \), \( M_\lambda \) is continuous at \( x \). Again, \( x \) was arbitrary and so \( M_\lambda \) is continuous.

According to the previous ideas, each \( U \) is open if and only if all of its translates \( U + a \) are open. Consequently, the topology is characterized by \( \mathcal{U} = \mathcal{U}(0) \).

2.1.5 Definition. (a) A filterbase \( \mathcal{B} \subset \mathcal{U} \) is called a local base if each \( U \in \mathcal{U} \) contains a \( B \in \mathcal{B} \).

(b) A set \( C \) is convex if for all \( a, b \in C \), we have \( \lambda a + (1 - \lambda)b \in C \) for all \( \lambda \in [0, 1] \).

(c) A set \( B \subset X \) is bounded if for each \( U \in \mathcal{U} \) there is \( s > 0 \) such that for all \( t > s, B \subseteq tU \).

(d) A metric on \( X \) is called invariant if for all \( x, y, z \in X \),

\[
d(x + z, y) = d(x, y).
\]

2.1.6 Definition. A topological vector space is called

(a) locally convex if it has a local base of convex sets.

(b) locally bounded if 0 has a bounded neighborhood.

(c) locally compact if 0 has a compact neighborhood.

(d) metrizable if the topology is induced by a metric.

(e) an F-space if the topology is induced by an invariant metric.

(f) a Fréchet space if \( X \) is a locally convex F-space.

(g) normable if the topology on \( X \) comes from a norm.

2.1.7 Examples. 1. Let \( L^p([0, 1]) \), \( 0 < p < 1 \) be the space of measurable functions \( f : [0, 1] \to \mathbb{R} \) such that \( \int_0^1 |f(x)|^p dx < \infty \), with functions equal almost everywhere identified. The function \( d(f, g) = \int_0^1 |f(x) - g(x)|^p dx \) is a metric on \( L^p([0, 1]) \). With the inherited metric topology, \( L^p([0, 1]) \), \( 0 < p < 1 \) is not a locally convex topological vector space. To see this, consider any open ball around 0, i.e.,

\[
\left\{ f \in L^p([0, 1]) : \int_0^1 |f(x)|^p dx < R \right\}.
\]
Given $\epsilon > 0$ and $n \geq 1$, we select $n$ disjoint intervals in $[0, 1]$ (not necessarily covering $[0, 1]$), say $I_1, \ldots, I_n$, and we set

$$f_k(x) = \left(\frac{\epsilon}{\mu(I_k)}\right)^{-p} \chi_{I_k}(x), \quad k = 1, \ldots, n,$$

where $\mu$ is considered to be the Lebesgue measure. Then

$$\int_0^1 |f_k(x)|^p dx = \epsilon,$$

and so every $f_k$ is at distance $\epsilon$ from 0. However, since the $f_k$’s are supported on disjoint intervals, their average

$$g_n(x) = \frac{1}{n} \sum_{k=1}^n f_k(x)$$

satisfies

$$\int_0^1 |g_n(x)|^p dx = \frac{1}{n^p} \sum_{k=1}^n \int_0^1 |f_k(x)|^p dx = n^{1-p} \epsilon.$$

Since $1 - p > 0$, the distance between $g_n$ and 0 can be made arbitrarily large with a suitable choice of $n$. In fact, what this means is that the only convex open set in $L^p([0, 1])$ is the whole space.

However, $L^p([0, 1])$ is locally bounded and an $F$-space, since and it admits a complete translation invariant metric with respect to which the vector space operations are continuous.

2. On the other hand, the spaces $L^p(\mu)$ for $p \geq 1$ have their metric coming from a norm and so they are locally convex.

We will see later that a topological vector space is normable if and only if it is locally bounded and locally convex. Also, $X$ is locally compact and normable if and only if $\dim X < \infty$. 