Functional Analysis, Math 7320
Lecture Notes from September 20, 2016

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1.6.52 Lemma. *Let (X, τ) be a topological space and let $Y \subset X$.*

(a) Y is quasi-compact with respect to the trace topology if and only if each open cover of Y in X has a finite subcover.

(b1) If X is Hausdorff and Y is compact, then Y is closed.

(b2) If X is compact and Y is closed, then Y is compact.

(c) If X is normed and Y is compact, then Y is bounded.

Proof. (a) Proven in the previous lecture.

(b1) We will show the complement $X \setminus Y$ is open in X. To this end, let $x \in X \setminus Y$. Now for each $y \in Y$, since X is Hausdorff, there are disjoint open sets U_y and V_y containing x and y, respectively. We then have $Y \subset \bigcup_{y \in Y} V_y$, and since $\{V_y\}_{y \in Y}$ is an open cover and Y is compact, there is a finite subcover $(V_y)_{y \in F}$ of Y with $|F| < \infty$ and

$$\bigcup_{y \in F} V_y \cap \left(\bigcap_{y \in F} U_y \right) = \emptyset,$$

since every V_y is disjoint from U_y. But as a finite intersection of open sets, $\bigcap_{y \in F} U_y$ is open and contains x. So $\bigcap_{y \in F} U_y \subset X \setminus Y$, which implies $X \setminus Y$ is open in X.

(b2) Let X be compact and let Y be closed. Then $X \setminus Y$ is open in X and given an open cover $(U_i)_{i \in I}$ of Y, composed by open sets in (X, τ), we see that $\left(\bigcup_{i \in I} U_i\right) \cup (X \setminus Y)$ is an open cover of X. Now since X is compact, there is a finite subcover of X which also covers Y. If $X \setminus Y$ is an element of this finite subcover, then by removing it, we have found a finite subcover of Y. We infer from this property that (Y, τ_Y) has the Heine-Borel/finite subcover property as well, because every set $V \subset Y$ that is open in τ_Y is obtained from $V = Y \cap U$ with U open in X. Thus, after "lifting" the open cover with respect to τ_Y to an open cover in X and passing to a finite subcover, intersecting the sets with Y again gives the desired finite open subcover in (Y, τ_Y).

It remains to show the Hausdorff property to establish compactness. Given $x, y \in Y$ with $x \neq y$, there are open (in X) sets V_x and V_y with $x \in V_x$, $y \in V_y$ and $V_x \cap V_y = \emptyset$. Thus, $V_x \cap Y$ and $V_y \cap Y$ have the desired separation properties to make (Y, τ_Y) Hausdorff. Hence (Y, τ_Y) is compact.

(c) In a normed space, if $(B_n(0))_{n\in\mathbb{N}}$ is an open cover of X, then it also covers Y. Thus there is a finite subcover $(B_n(0))_{n\in F}$ of Y with $|F| < \infty$, and by choosing $N = \max\{n \in F\}$ we have

$$Y \subset B_N(0) \subset \overline{B}_N(0),$$

which means Y is bounded.

□

2 Topological Vector Spaces

2.1 Fundamental properties

2.1.1 Definition. A vector space X together with a topology τ is called a *topological vector space* if

1. for every point $x \in X$, the singleton $\{x\}$ is a closed set.
2. the vector space operations

$$+ : X \times X \to X, \quad (x, y) \mapsto x + y$$

and

$$\cdot : \mathbb{K} \times X \to X, \quad (\lambda, x) \mapsto \lambda x$$

are continuous with respect to the product topology on $X \times X$ and $\mathbb{K} \times X$, respectively.

2.1.2 Theorem. *Every normed space is a topological vector space.*

Proof. As shown before, '$+$' and '$\cdot$' are continuous operations. Moreover,

$$\bigcap_{n\in\mathbb{N}} \overline{B}_{\frac{1}{n}}(x) = \{x\}$$

is closed as an intersection of closed sets. □

2.1.3 Remark. (a) For $a, b \in X$, let $V_a \in \mathcal{U}(a)$ and $V_b \in \mathcal{U}(b)$ be open sets. Since each neighborhood of $(a, b) \in X \times X$ contains $V_a \times V_b$, continuity of '$+$' means that for each $U \in \mathcal{U}(a + b)$, we can find V_a, V_b as above with

$$V_a + V_b = \{a' + b' : a' \in V_a, b' \in V_b\} \subset U.$$

(b) Analogously, since $\lambda \in \mathbb{K}$ and $\mathbb{K}$ is equipped with the topology of open balls, for $U \in \mathcal{U}(\lambda x)$, there is an open $V_x \in \mathcal{U}(x)$ and $\delta > 0$ such that

$$B_\delta(\lambda)V_x = \{\lambda' x' : \lambda' \in B_\delta(\lambda), x' \in V_x\} \subset U.$$

Next, we explore implications of continuity for the topological structure of the space.

2.1.4 Theorem. *Let X be a topological vector space, $a \in X$ and $\lambda \in \mathbb{K}$ with $\lambda \neq 0$. Then both the translation operator $T_a : X \to X$ with $T_a x = x + a$ and the scaling operator $M_\lambda : X \to X$ with $M_\lambda x = \lambda x$ are homeomorphisms of X onto X.*

Proof. For $a \in X$ and $0 \neq \lambda \in \mathbb{K}$, we note that $T_{-a} \circ T_a = \text{id}$ and that $M_{\lambda^{-1}} \circ M_\lambda = id$, so T_a and M_λ are $1-1$. Hence, it is sufficient to show that for each $a \in X$ and $0 \neq \lambda \in \mathbb{K}$, T_a and M_λ are continuous. By the continuity of '+', given $x, a \in X$, then for $U \in \mathcal{U}(x+a)$ there are $V_a \in \mathcal{U}(a)$, $V_x \in \mathcal{U}(x)$ such that $V_a + V_x \subset U$ and hence $a + V_x \subset U$. This means $T_a(V_x) \subset U$, and so T_a is continuous at x and since x was arbitrary, T_a is continuous.

Similarly, given $U \in \mathcal{U}(\lambda x)$, there is V_x and $\delta > 0$ such that $B_\delta(\lambda)V_x \subset U$, which means $\lambda V_x \subset U$, and thus for each $\lambda \neq 0$, M_λ is continuous at x. Again, x was arbitrary and so M_λ is continuous. □

According to the previous ideas, each U is open if and only if all of its translates $U + a$ are open. Consequently, the topology is characterized by $\mathcal{U} \equiv \mathcal{U}(0)$.

2.1.5 Definition. (a) A filterbase $\mathbb{B} \subset \mathcal{U}$ is called a *local base* if each $U \in \mathcal{U}$ contains a $B \in \mathbb{B}$.

(b) A set C is *convex* if for all $a, b \in C$, we have $\lambda a + (1-\lambda)b \in C$ for all $\lambda \in [0,1]$.

(c) A set $B \subset X$ is *bounded* if for each $U \in \mathcal{U}$ there is $s > 0$ such that for all $t > s$, $B \subset tU$.

(d) A metric on X is called *invariant* if for all $x, y, z \in X$,

$$d(x+z, y) = d(x, y).$$

2.1.6 Definition. A topological vector space is called

(a) *locally convex* if it has a local base of convex sets.

(b) *locally bounded* if 0 has a bounded neighborhood.

(c) *locally compact* if 0 has a compact neighborhood.

(d) *metrizable* if the topology is induced by a metric.

(e) an *F-space* if the topology is induced by an invariant metric.

(f) a *Fréchet space* if X is a locally convex F-space.

(g) *normable* if the topology on X comes from a norm.

2.1.7 Examples. 1. Let $L^p([0,1])$, $0 < p < 1$ be the space of measurable functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(x)|^p dx < \infty$, with functions equal almost everywhere identified. The function $d(f,g) = \int_0^1 |f(x) - g(x)|^p dx$ is a metric on $L^p([0,1])$. With the inherited metric topology, $L^p([0,1])$, $0 < p < 1$ is not a locally convex topological vector space. To see this, we consider any open ball around 0, i.e.,

$$\left\{ f \in L^p([0,1]) : \int_0^1 |f(x)|^p dx < R \right\}.$$

Given $\epsilon > 0$ and $n \geq 1$, we select n disjoint intervals in $[0,1]$ (not necessarily covering $[0,1]$), say $I_1, \ldots, I_n$, and we set

$$f_k(x) = \left(\frac{\epsilon}{\mu(I_k)}\right)^{-p} \chi_{I_k}(x), \quad k = 1, \ldots, n,$$

where μ is considered to be the Lebesgue measure. Then

$$\int_0^1 |f_k(x)|^p dx = \epsilon,$$

and so every f_k is at distance ϵ from 0. However, since the f_k's are supported on disjoint intervals, their average

$$g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f_k(x)$$

satisfies

$$\int_0^1 |g_n(x)|^p dx = \frac{1}{n^p} \sum_{k=1}^{n} \int_0^1 |f_k(x)|^p dx = n^{1-p}\epsilon.$$

Since $1-p > 0$, the distance between g_n and 0 can be made arbitrarily large with a suitable choice of n. In fact, what this means is that the only convex open set in $L^p([0,1])$ is the whole space.

However, $L^p[(0,1)]$ is locally bounded and an F-space, since and it admits a complete translation invariant metric with respect to which the vector space operations are continuous.

2. On the other hand, the spaces $L^p(\mu)$ for $p \geq 1$ have their metric coming from a norm and so they are locally convex.

We will see later that a topological vector space is normable if and only if it is locally bounded and locally convex. Also, X is locally compact and normable if and only if $\dim X < \infty$.