Functional Analysis, Math 7320
Lecture Notes from October 04, 2016

taken by Qianfan Bai

Last Time

- Nice local bases
- Balanced neighborhoods
- Linear functionals, continuity and null space

Recall the theorem from the previous lecture:

2.2.0 Theorem. *Let Λ be a nontrivial linear functional on a topological vector space X and there is $x \in X$ with $\Lambda x \neq 0$, then the following are equivalent:*

(a) Λ is continuous.

(b) $\mathcal{N}(\Lambda)$ is closed.

(c) $\mathcal{N}(\Lambda)$ is not dense in X.

(d) Λ is bounded on a neighborhood V of 0.

Proof. We had (a)⇒(b)⇒(c)⇒(d).
To show (d)⇒(a), we only need to show continuity at 0, because then by linearity, Λ is continuous everywhere, hence continuous.
Assume there is $M > 0$ and a neighborhood V of 0 such that

$$|\Lambda x| < M$$

for $x \in V$. Given $\epsilon > 0$, then we choose $W := \frac{\epsilon}{M} V$, then by linearity, for $x \in W$,

$$|\Lambda x| < \frac{\epsilon}{M} M = \epsilon.$$

This finishes the proof.

□

2.2.1 Exercise. The subspace c and c_0 are closed subspaces of l^∞ (and hence are Banach spaces). The space c_{00} is only a subspace in c_0, but not closed in c_0 (and hence not in l^∞).

$$c_0 := \{(x_j)_{j\in\mathbb{N}} : \lim_{j\to\infty} x_j = 0\}$$

$$c_{00} = \{x \in l^\infty, \text{there is } n \in \mathbb{N} \text{ such that } x_j = 0 \text{ for all } j \geq n\}.$$

The closure of c_{00} is c_0.
Actually, c_{00} is dense in c_0, i.e. $\overline{c_{00}} = c_0 (\neq c_{00})$.
Let $x^{(k)} \in c_0$ be a sequence converging to $w \in l^\infty$. Take $\epsilon > 0$ and $N_0 \in \mathbb{N}$ such that

$$\sup_{1\leqslant j\leqslant\infty} |x_j^{(k)} - w_j| < \frac{\epsilon}{2}$$

for all $k > N_0$. For each k choose $N_1 \in \mathbb{N}$ such that

$$|x_j^{(k)}| \leqslant \frac{\epsilon}{2}$$

for all $j > N_1$.
Thus, $|w_j|$=$|w_j - 0| \leqslant |w_j - x_j^{(k)}|$+$|x_j^{(k)}| \leqslant \epsilon$ for $j > N_1$, and $k > N_0$, that means that the sequence w_j converges to 0 and $w \in c_0$. Hence c_0 is closed in l^∞.
A specific example of a Cauchy sequence in c_{00} that does not have a limit in c_{00} is the sequence $\{x_n\}$, where x_n is given by $x_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots)$. The vectors x_n do converge in l^∞ norm to the vector x= $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, but x does not belong to c_{00}.
While $\{x_n\}$ converges in c_0 and in l^∞, it does not converge in c_{00}. Since c_0 is complete, given a sequence in c_0 which is convergent in l^∞, it is Cauchy and hence convergent in c_0, so c_0 is closed. From c_{00} being dense in C_0, we conclude it is the closure of c_{00} in l^∞.

2.3 Finite-Dimensional (Sub)Spaces

2.3.2 Lemma. *Let Y be an n-dimensional subspace of a topological vector space (TVS) X, then*

(a) every vector-space isomorphism $f : \mathbb{K}^n \longrightarrow Y$ is a homeomorphism, and

(b) Y is closed.

Proof. (a) Take $S := \{x \in \mathbb{K}^n, \sum_{j=1}^n |x_j|^2 = 1\}$, then S is compact. Consider the map

$$f : \mathbb{K}^n \longrightarrow Y, \quad (c_j)_{j=1}^n \mapsto \sum_{j=1}^n c_j v_j,$$

with $\{v_j\}_{j=1}^n$ fixed basis in Y, we first show f is continuous:
Denote by $\{c_j\}$ the standard basis in $\mathbb{K}^n$ and set

$$v_j = f(c_j) \qquad j = 1, \ldots, n.$$

By linearity, for any $c = (c_1, \ldots, c_n)$,

$$f(c) = \sum_{j=1}^{n} c_j v_j.$$

The map $c \mapsto c_j$ is continuous and so are the addtion and scalar multiplication in Y. Thus, f is continuous.
We then have that $f(S) \equiv K$ is quasi-compact. Since Y inherits the Hausdorff property of X, K is compact.
Next, $f(0) = 0$, and we know $0 = f(0) \notin K$ by linear independence of $\{v_j\}_{j=1}^n$. Take balanced neighborhood V of 0 in X such that $V \cap K = \emptyset$, then

$$E = f^{-1}(V) = f^{-1}(V \cap Y)$$

does not intersect S. By linearity, E is balanced (in $\mathbb{K}^n$), and hence connected. So $E \subset B_1^{\mathbb{K}^n}(0)$. Thus, f^{-1} is a bounded linear map, hence continuous.
We conclude that f is homeomorphism.

(b) For closedness, let $p \in \overline{Y}$, and choose f and V be as above.
By Theorem 11.5.6 given in previous notes, we know there is $t > 0$ such that $p \in tV$.
Since $p \in tV$ and $p \in \overline{Y}$ implies that for every open set $U \ni x$, $U \cap tV$ is open and contains x, and then by the characterization of the closure, $U \cap tV \cap Y \neq \emptyset$, so

$$p \in \overline{Y \cap (tV)}.$$

Now using the choice of tV so that $tV \cap Y = f(f^{-1}(tV \cap Y)) \subset f(tB_1^{\mathbb{K}^n}(0))$,

$$p \in \overline{Y \cap (tV)} \subset \overline{f(tB_1^{\mathbb{K}^n}(0))} \subset \overline{f(t\overline{B_1^{\mathbb{K}^n}(0)}} = f(t\overline{B_1^{\mathbb{K}^n}(0)}) \subset Y\,.$$

The last identity is because f^{-1} is continuous, so f maps closed sets to closed sets. Hence, Y is closed. □

2.3.3 Lemma. *Let $Y \subset X$ be a locally compact sub-vector space of a TVS X, then Y is closed and finite-dimensional in X.*

Proof. By definition of local compactness, (Y, τ_Y) has a compact neighborhood K of 0. By an earlier theorem, K is bounded. Thus, $(\frac{1}{n}K)_{n\in\mathbb{N}}$ is a local base.
Let V be a balanced open neighborhood in K^0. We know $\bigcup_{x\in K}(x + \frac{1}{2}V)$ is an open cover of K, so there exist $x_1, x_2, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{j=1}^{m}(x_j + \frac{1}{2}V)$$

Take $Z = Span\{x_j\}_{j=1}^m$, so dim $Z \leq m$, and by the preceding lemma (Lemma 2.3.1), Z is closed. Replacing each x_j by Z, we get

$$V \subset K \subset Z + \frac{1}{2}V$$

Since $\lambda Z = Z$ for every scalar $\lambda \neq 0$, we have $\frac{1}{2}Z = Z$. Scaling by $\frac{1}{2}$ gives

$$\frac{1}{2}V \subset Z + \frac{1}{4}V$$

which implies

$$V \subset Z + \frac{1}{2}V \subset Z + Z + \frac{1}{4}V = Z + \frac{1}{4}V.$$

Continuing in this manner, we get

$$V \subset Z + \frac{1}{2^n}V$$

for each $n \in \mathbb{N}$ and from $V \subset \bigcap_{n\in\mathbb{N}}(Z + \frac{1}{2^n}V)$, using the local base property of $(\frac{1}{2^n}V)_{n\in\mathbb{N}}$, we have $V \subset \overline{Z}$.
Next, Z is finite - dimensional, so $\overline{Z} = Z$, so $V \subset Z = \frac{1}{k}Z$ for any $k \in \mathbb{N}$, or equivalently, $kV \subset Z$.
Taking the union over all kV gives by the exhausting property of balanced neighborhoods

$$Z = \bigcup_{k\in\mathbb{N}}(kV) = Y.$$

Thus, dim $Y \leq m$ and Y is closed. □

2.4 Seminorms and local convexity

In this section we will see that locally convex TVS are characterized as topologies given by seminorms.
We recall the characterization of the initial topology with respect to $(f_i)_{i\in I}$, $f_i : X \to X_i$ and the product topology on $\prod_{i\in I} X_i$, such that

$$\eta : X \to \prod_{i\in I} X_i, \quad x \mapsto (f_i(x))_{i\in I}$$

is continuous.
We say that η or $(f_i)_{i\in I}$ *separates* points of X if η is 1-1.
In that case, we can identify X with $\eta(X)$ and then the initial topology τ_X is the trace topology on $\eta(X)$ in $\prod_{i\in I} X_i$.