Functional Analysis, Math 7320
Lecture Notes from October 11, 2016

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2.8 Warm Up

2.8.1 Problem. Let $\mathcal{X}$ be a Topogical Vector Space and $\mathcal{Y}$ a closed subspace, and $\mathcal{F}$ a finite dimentional subspace. Show $\mathcal{Y} + \mathcal{F}$ is closed.

Proof. Note that the Quoient topology is the final topology on the quotient space $\mathcal{X}/\mathcal{Y}$ with respect to the Quotient map q:$\mathcal{X} \to \mathcal{X}/\mathcal{Y}$, thus q is continuous. Recall that in the quotient topology equivalence classes are defined by $[x] = \{x + y : y \in \mathcal{Y}\}$, thus $q(x) = x + \mathcal{Y}$. Given α from the field $\mathbb{F}$ from our Topogical Vector Space $\mathcal{X}$, and $x, z \in \mathcal{X}$, then we have

$$\alpha q(x) = \alpha(x + \mathcal{Y}) = \alpha[x] = \{\alpha x + \alpha y : y \in \mathcal{Y}\} = \{\alpha x + y : y \in \mathcal{Y}\} = [\alpha x] = q(\alpha x)$$

$$q(x + z) = x + z + \mathcal{Y} = [x + z] = \{x + z + y : y \in \mathcal{Y}\} = \{x + z + 2y : y \in \mathcal{Y}\} =$$
$$\{\alpha x + \alpha y : y \in \mathcal{Y}\} + \{\alpha z + \alpha y : y \in \mathcal{Y}\} = [x] + [z] = q(x + z)$$

Thus q is a linear map. Map $\mathcal{F}$ to q($\mathcal{F}$), and observe that q($\mathcal{F}$) is a finite dimentional vector space and is closed by the linearity of q and the closedness of finite dimentional subspaces. As [0] $= \{y : y \in \mathcal{Y}\}$, then q$^{-1}(x) = x + \mathcal{Y}$ and q^{-1}(q($\mathcal{F}$)) $= \mathcal{Y} + \mathcal{F}$. Lastly, recall that if q is continuous then the preimage of a closed set is close hence $\mathcal{Y} + \mathcal{F}$ is closed .

□

Notice how our tools make this proof so easy. Next we see how certain topologies imply the existence of seminorms.

2.8.2 Theorem. *Let A be an absorbing set and μ_A the minkowski functional, the we have the following results:*

(a) For any $\lambda \geq 0$ and $x \in \mathcal{X}$, $\mu_A(\lambda x) = \lambda \mu_A(x)$.

(b) If A is convex, then μ_A is subadditive.

(c) If A is convex and balanced, then μ_A is a seminorm.

(d) If B := $\{x \in \mathcal{X} : \mu_A < 1\}$, and C := $\{x \in \mathcal{X} : \mu_A \leq 1\}$, then B ⊂ A ⊂ C, and $\mu_A = \mu_B = \mu_C$.

(e) If p is a seminorn on $\mathcal{X}$, then p = μ_A with A = $\{\ x \in \mathcal{X} : p(x) < 1\}$.

(f) If $B \subset X$ is bounded, so is $\overline{B}$.

Proof. (a) Recall that a set A is absorbing if for every $x \in \mathcal{X}$ there exist a t> 0 such that $x \in tA$. In particular if $x = 0$ then $0 \in tA$ implies there exists an $a \in A$ and a t such that $at = 0$ thus $a = 0$, and $0 \in A$. Given $\lambda \geq 0$, $\mu_A(\lambda x) =$inf$\{t > 0 : \lambda x \in tA\} = \lambda$inf$\{t > 0 : x \in tA\} = \lambda\mu_A(x)$.

(b) We want to show for x,y $\in \mathcal{X}$ that, $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$. Let $\epsilon > 0$, $t = \mu_A(x) + \epsilon$, $s = \mu_A(y) + \epsilon$. Note, $\frac{x}{t} \in$ A and $\frac{y}{s} \in A$ by defintion of μ_A. $\frac{t}{s+t}\frac{x}{t} + \frac{s}{s+t}\frac{y}{s} = \frac{x+t}{s+t} \in$ A by convexity. Thus $\mu_A(x+y) \leq s + t = \mu_A(x) + \mu_A(y) + 2\epsilon$. This is true for every $\epsilon > 0$, hence $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$.

(c) μ_A is a seminorm if $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$ and $\mu_A(\alpha x) = |\alpha|\mu_A(x)$, for all $x, y \in \mathcal{X}$ and scalers α. The first part follows from part b. By assumption, A is convex absorbing and balanced so given t $\in \mathbb{K}$ we let t $= |t| + \alpha$ with $|\alpha| = 1$. Balacedness then implies $\mu_A(tx) = |t|\mu_A(\alpha x) = |t|\mu_A(x)$.

(d) B $\subset$ A $\subset$ C clearly gives $\mu_C \leq \mu_A \leq \mu_B$. To show equality let $x \in \mathcal{X}$, choose s and t such that $\mu_C(x) < s < t$, then $\frac{x}{s} \in C$ so $\mu_A(\frac{x}{s}) \leq 1$ by defintion of C. Thus $\mu_A(\frac{s}{t}\frac{x}{s}) \leq \frac{s}{t} < 1$ so $\frac{x}{t} \in B$ and $\mu_B(x) \leq t$.

This works for any s , t with $t > s > \mu_C(x)$ so $\mu_B(x) \leq \mu_C(x)$. This implies $\mu_C = \mu_A = \mu_B$.

(e) Consider A as defined by p, this A is Balanced. We also see A is convex if $x, y, \in$ A, $0 < t < 1$, then $p(tx + (1-t)y) \leq tp(x) + (1-t)p(y)$ we also know A is absorbing.

If $x \in \mathcal{X}$ and $s > p(x)$, then $p(\frac{x}{s}) = \frac{1}{s}p(x) < 1$. So $\mu_A(x) \leq s$, and thus $\mu_A \leq p$.

On the other hand if $0 < t \leq p(x)$, then $p(\frac{x}{t}) \geq 1$ and $t^{-1}x \notin$ A. By the balancedness of A, p(x)$\leq \mu_A(x)$. Lastly as A is absorbing we know $\mu_A(x)$ is finite for all $x \in \mathcal{X}$. We conclude p $= \mu_A$.

□

2.8.3 Remark. We can convert between seminorms and minkowsiki functionals of convex balaced open neighborhoods.

2.8.4 Theorem. *Let $\mathbb{B}$ be a convex balanced local open base in a Topological Vector Space $\mathcal{X}$ then,*

(a) If V $\in \mathbb{B}$, then V $= \{x \in \mathcal{X} : \mu_V(x) < 1\}$

(b) $(\mu_V)_{V \in \mathbb{B}}$ is a family of continous seminorms that seperates points on $\mathcal{X}$.

Proof. (a) If $x \in$ V, then by V being open and continuity of $(\alpha, x) \mapsto \alpha x$. We have $\frac{x}{t} \in V$ for some t < 1. Thus $\mu_V(x) < 1$. On the other hand, if $x \notin$ V and $\frac{x}{t} \in$ V then by the balancedness of V, if $|\frac{1}{t}| \leq 1$ implies $\frac{1}{t}V \subset V$ and $x \in V$ thus $t > 1$. We conclude in this case $\mu_V(x) \geq 1$.

(b) Next, μ_V is a seminorm by V $\in \mathbb{B}$ using the preceding part, we set r $>$ 0, $x, y \in \mathcal{X}$ from the triangle inequality, $|\mu_V(x) - \mu_V(y)| \leq \mu_V(x-y)$ and if $x - y \in$rV scaling gives $\mu_V(x-y)$ <r. As $r > 0$ is the radius of the ball that characterizes the topology, then μ_V is continous. Moreover if $x \in \mathcal{X}\backslash\{0\}$, then there is V$\in \mathbb{B}$ such that $x \notin V$. So $\bigcap_{V\in\mathbb{B}} = \{0\}$, and thus μ_V separate points.

□

Next we characterize the topology induced by these seminorms.

2.8.5 Lemma. *A seminorm p on a Topological Vector Space $\mathcal{X}$ is continuous, iff V(p,1)* $= \{x \in \mathcal{X} : p(x) < 1\}$ *is a bounded neighborhood of 0.*

Proof. If p is continuous this is true by definition. Conversely suppose V(p,1) is a neighborhood of 0. By scaling with $r > 0$, rV(p,1) $= \{x \in \mathcal{X} : p(x) < r\}$. Which gives a local base , obtained as rV(p,1) $= p^{-1}(B_r(0))$. Moreover, given $\epsilon > 0$, $x \in \mathcal{X}$ and r = p(x), then if $q < \epsilon$, we have p(x + qV(p,1)) $\subset$ r $+B_\epsilon(0)$. Thus p is continuous at each $x \in \mathcal{X}$. □

2.8.6 Theorem. *Let $\mathcal{P}$ be a family of seminorms on a vector space $\mathcal{X}$ that separates points, and V(p,$\frac{1}{n}$)* $= \{x : p(x) < \frac{1}{n}\}$. *Let $\mathbb{B}$ be the collection $\mathbb{B} = \{V = \bigcap_{i=1}^{n} V(p_i, \frac{1}{n}\}$, then $\mathbb{B}$ is a convex balanced local base for a topology $\mathcal{T}$ on $\mathcal{X}$ which makes $\mathcal{X}$ a locally convex Topological Vector Space, and*

(a) Each p $\in \mathcal{P}$ is continous.

(b) $E \subset \mathcal{X}$ is bounded iff each p is bounded on E.