1 Last time

Locally convex TVS as spaces whose topology is induced by a family of seminorms.

2 Warm-up

In a TVS, given $V \in \mathcal{U}$ there is $C \in \mathcal{U}$ s.t. $C \subset V$ and $C = C$.

Proof. We know by continuity of addition at $(0,0)$ that, there is a $W \in \mathcal{U}$ with $W + W \subset V$. We also know that, $W = \bigcap_{Y \in \mathcal{U}} (W + Y)$. So, by choosing $Y = W$, we have $W \subset W + W \subset V$. □

3 Metrization

3.0.1 Question. When does the topology of a TVS come from a metric?

We have seen that in a TVS which comes from a metric, we have a countable local base, i.e. $(B_{1/n}(0))_{n \in \mathbb{N}}$. The existence of such a local base is also sufficient to obtain $\tau$ from a metric.

3.0.2 Theorem. Let $X$ be a TVS with a countable local base. Then, there is a metric $d$ s.t.

a) Every open set in $\tau$ is the union of open balls w.r.t. $d$.

b) Each open $B_r(0)$, $r > 0$, is balanced.

c) $d$ is translation invariant.

d) If $X$ is locally convex, then $d$ can be chosen so that all $B_r(0)$ are convex.

Proof. Consider a countable local base $B'$. Let $B$ be a countable local base s.t. $B = (V_n)_{n \in \mathbb{N}}$ with each $V_n$ balanced and $V_{n+1} + V_{n+1} + V_{n+2} + V_{n+1} \subset V_n$ (using continuity of addition). We then have for all $n, k \in \mathbb{N}$

$V_{n+1} + V_{n+2} + \ldots + V_{n+k} \subset V_n$ because $V_{n+k-1} + V_{n+k} \subset V_{n+k-1} + V_{n+k}$ and

$V_{n+k-2} + V_{n+k-1} + V_{n+k} \subset V_{n+k-2} + V_{n+k-1} + V_{n+k}$

$\subset V_{n+k-2} + V_{n+k-1} + V_{n+k} \subset V_{n+k-2} + V_{n+k-1} + V_{n+k}$

$\subset V_{n+k-2} + V_{n+k-1} + V_{n+k} \subset V_{n+k}$
We can iteratively remove tail terms and end up with

\[ V_{n+1} + V_{n+2} + V_{n+2} + V_{n+2} \subset V_{n+1} + V_{n+1} \subset V_n \]

Consider the dyadic rationals \( D = \{ \sum_{j=1}^{n} \frac{c_j}{2^j}, \ c_j \in \{0, 1\}, n \in \mathbb{N} \} \) and let

\[ \phi : D \cup [1, \infty] \rightarrow P(X) \]

\[ \phi(r) = \begin{cases} X & r \geq 1 \\ c_1(r)V_1 + c_2(r)V_2 + ... + c_n(r)V_n, & r \in D \end{cases} \]

From the construction of \( (V_n)_{n \in \mathbb{N}} \),

\[ \phi(\sum_{j=n_1}^{n_2} \frac{c_j}{2^j}) = \sum_{j=n_1}^{n_2} c_jV_j \subset V_{n_1-1}. \]

Further, let \( f : X \rightarrow \mathbb{R} \)

\[ f(x) = \inf \{ r : x \in \phi(r) \} \]

and let \( d(x, y) = f(y - x) \).

We first consider properties of \( \phi \). For \( r, s \in D \), we claim that \( \phi(r) + \phi(s) \subset \phi(r + s) \).

(i) If \( r + s \geq 1 \) then the RHS gives \( \phi(r + s) = X \), and there is nothing to show.

(ii) Next, let \( r + s \in D \).

(Case I) \( c_n(r) + c_n(s) = c_n(r + s) \) for all \( n \). Then

\[ \phi(r + s) = \sum_{j=1}^{N} c_j(r + s)V_j \]

\[ = \sum_{j=1}^{N} c_j(r)V_j + \sum_{j=1}^{N} c_j(s)V_j \]

\[ = \phi(r) + \phi(s) \]

(Case II) There is an \( n \in \mathbb{N} \) s.t. \( c_n(r) + c_n(s) \neq c_n(r + s) \). Let \( N \) be the smallest index for which this occurs, then \( c_N(r) = c_N(s) = 0 \), and \( c_N(r + s) = 1 \). Consequently,

\[ \phi(r) = c_1(r)V_1 + c_2(r)V_2 + ... + c_{N-1}(r)V_{N-1} + 0.V_N \]

\[ + c_{N+1}(r)V_{N+1} + c_{N+2}(r)V_{N+2} + c_{N+3}(r)V_{N+3} + \ldots \]

\[ \subset c_1(r)V_1 + ... + c_{N-1}(r)V_{N-1} + V_{N+1} + V_{N+1} \]
We conclude that 
\[ \phi(s) \subset c_1(s)V_1 + \ldots + c_{N-1}(s)V_{N-1} + V_{N+1} + V_{N+1} \]
Hence, 
\[ \phi(r) + \phi(s) \subset c_1(r+s)V_1 + \ldots + c_{N-1}(r+s)V_{N-1} + V_{N+1} + V_{N+1} + V_{N+1} \]
\[ \subset c_1(r+s)V_1 + \ldots + c_{N-1}(r+s)V_{N-1} + c_N(r+s)V_N \]
\[ \subset \phi(r+s) \]

Next, we observe that for \( r \in D \cup [1, \infty), 0 \in \phi(r) \), because \( \phi(r) \) contains at least one neighborhood of 0. Moreover, \( \{ \phi(r) : r \in D \} \) is totally ordered by set inclusion, because if \( r < t \), then \( \phi(r) \subset \phi(r) + \phi(t - r) \subset \phi(t) \). From the definition of \( f \), this implies for all \( x, y \in X \), \( f(x + y) \leq f(x) + f(y) \), as we see below:

From the range of \( f \) being \([0, 1]\), assume \( RHS \leq 1 \):

Fix \( \epsilon > 0 \), then there are \( r, s \in D \) with \( f(x) < r \), \( f(y) < s \) and \( r + s < f(x) + f(y) + \epsilon \). Ordering implies \( x \in \phi(r), y \in \phi(s) \) and therefore \( x + y \in \phi(r) + \phi(s) \subset \phi(r+s) \). Therefore \( f(x + y) \leq r + s < f(x) + f(y) + \epsilon \) This is true for any \( \epsilon > 0 \), so \( f(x + y) \leq f(x) + f(y) \).

Next, by balancedness of each \( V_j \), \( f(x) = f(-x) \), \( f(0) = 0 \), and \( f(x) > 0 \) for \( x \neq 0 \).

This results from each \( \phi(r) \) being balanced, and \( f(0) = 0 \) because \( 0 \in \phi(r) \) for each \( r \in D \), and if \( x \neq 0 \), then there is \( V_N \) with \( x \not\in V_N \) (since every TVS is a Hausdorff space, so we can separate \( x \) from 0 by two disjoint open neighborhoods), since by construction \( V_k \supset V_N \) for all \( k \leq N \) therefore, there is \( s \) s.t. \( x \not\in \phi(r) \) for all \( r < s \), and so by definition \( f(x) \geq 0 \).

We conclude that \( d(x, y) = f(x - y) \) defines a (translationally) invariant metric. To see that \( d \) is compatible with \( \tau \), consider

\[ B_\delta(0) = \{ x : f(x) < \delta \} \]
\[ = (\text{by total ordering}) \]
\[ = \bigcup_{r < \delta} \phi(r) \]
\[ = \bigcup_{r \in D, r < \delta} \phi(r) \]

Thus, \( B_{\frac{1}{2^n}}(0) \) is a local base.

If each \( V_n \) is convex, so is each \( \phi(r) \) and hence \( B_{\frac{1}{2^n}}(0) \)

\( \square \)
3.0.3 **Theorem.** Suppose that \((X, d_1)\) and \((Y, d_2)\) are metric spaces, and \((X, d_1)\) is complete. If \(E\) is closed in \(X\), \(f : E \to Y\) is continuous, and

\[
d_2(f(x'), f(x'')) \geq d_1(x', x'')
\]

for all \(x', x'' \in E\), then \(f(E)\) is closed.

**Proof.** Pick \(y \in f(E)\). There exist points \(x_n \in E\) so that \(y = \lim f(x_n)\). Thus \(\{f(x_n)\}\) is Cauchy in \(Y\). Our hypothesis \((*)\) implies therefore that \(\{x_n\}\) is Cauchy in \(X\). Being a closed subset of a complete metric space, \(E\) is complete; Hence there exists \(x = \lim x_n\) in \(E\). Since \(f\) is continuous, \(f(x) = \lim f(x_n) = y\).

Thus \(y \in f(E)\). \(\square\)

3.0.4 **Theorem.** (a) If \(d\) is a translation-invariant metric on a v.s. \(X\), then

\[
d(nx, 0) \leq nd(x, 0)
\]

(b) If \(\{x_n\}\) is a sequence in a metrizable t.v.s. \(X\) and if \(x_n \to 0\) as \(n \to \infty\), then there are positive scalars \(\gamma_n\) s.t. \(\gamma_n \to \infty\) and \(\gamma_n x_n \to 0\).

**Proof.** Statement (a) follows from triangle inequality plus translation invariance of the metric

\[
d(0, nx) \leq d(0, x) + d(x, 2x) + d(2x, 3x) + \ldots + d((n-1)x, nx)
\]

\[
\leq \sum_{k=1}^{n} d(kx, (k-1)x)
\]

\[
= d(x, 0) + d(x, 0) + d(x, 0) + \ldots + d(x, 0)
\]

\[
= nd(x, 0)
\]

To prove (b), let \(d\) be a metric as in (a), compatible with the topology of \(X\). Since \(d(x_n, 0) \to 0\), there is an increasing sequence of positive integers \(n_k\) such that \(d(x_n, 0) < \frac{1}{k^2}\) if \(n \geq n_k\). Put

\[
\gamma_n = \begin{cases} 
1 & \text{if } n < n_1, \\
k & \text{if } n_k \leq n < n_{k+1},
\end{cases}
\]

for such \(n\) we have

\[
d(\gamma_n x_n, 0) = d(kx_n, 0) \leq kd(x_n, 0) < \frac{1}{k^2}.\]

Hence \(\gamma_n x_n \to 0\) as \(n \to \infty\). \(\square\)