3.2.14 Corollary. Let $X$ be $F$-spaces, $Y$ be topological vector spaces and let $\Gamma$ be a collection of continuous linear maps from $X$ to $Y$. If
\[ B_x = \{Ax : A \in \Gamma\} \]
is bounded in $Y$, for every $x \in X$. Then $\Gamma$ is equicontinuous and hence uniformly bounded.

Proof. here we have
\[ B = \{x \in X : B_x \text{ is bounded}\} = X \]
is of second category, so the preceding theorem applies.

we specialize further to norm space $X, Y$.

3.2.15 Theorem. (Banach-Steinhaus) Let $\Gamma$ be a family of continuous linear mappings from a Banach space $X$ into a normed space $Y$. If for every $x \in X$,
\[ \sup_{A \in \Gamma} \|Ax\| < \infty, \]
then there exists $M > 0$ s.t for all $A \in \Gamma, x \in X, \|x\| \leq 1$, we have
\[ |Ax| \leq M. \]
Hence
\[ \sup_{A \in \Gamma} \|A\| \leq M. \]

we investigate consequence of sequence of operators:

Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of continuous map from $F$-space to Topological vector space $Y$, and for each $x \in X$,
\[ \lim_{n \to \infty} A_n(x) = A(x) \text{ exists.} \]

Question: is $A$ bounded?

3.2.16 Corollary. Let $X$ be $F$-spaces, $Y$ be topological vector spaces and let $(A_n)_{n \in \mathbb{N}}$ be a collection of continuous linear maps from $X$ to $Y$. And suppose for each $x \in X$,
\[ \lim_{n \to \infty} A_n(x) = A(x) \text{ exists.} \]
Then $A$ is linear and bounded.
**Proof.** Let \( x \in X \), by the condition of \((A_n x)_{n \in \mathbb{N}}\) in \( Y \), \((A_n x)_{n \in \mathbb{N}}\) is bounded in \( Y \), so \((A_n)_{n \in \mathbb{N}}\) is equicontinuous by Banach-Steinhaus for \( F \)-space.

Thus if \( W \in \mathcal{U}^Y \), then there is \( V \in \mathcal{U}^X \), s.t
\[
A_n(V) \subset W \text{ for each } n \in \mathbb{N}.
\]

then
\[
\left\{ y \in Y : y = \lim_{n \to \infty} A_n(x) \text{ for } x \in V \right\} \subset W.
\]

so
\[
A(V) \subset W.
\]

Hence \( A \) is bounded. \[\Box\]

**3.2.17 Example.** Consider \( c_{00} \), and let \( A_n : c_{00} \to c_{00} \) be given by:
\[
(A_n x)_m = \begin{cases} 
mx_m & \text{if } m \leq n \\
0 & \text{if } m > n
\end{cases}
\]

then
\[
\|A_n\| = n < \infty.
\]

So each \( A_n \) is bounded and for each \( x \in c_{00} \),
\[
\lim_{n \to \infty} A_n(x) = A(x) \text{ exists.}
\]

But, choosing \( e_m = (0, 0, \ldots, 0, 1, 0, \ldots) \) where 1 is in the \( m \)th position, shows:
\[
\sup_{\|x\|_{\infty} \leq 1, x \in c_{00}} \|Ax\| > \sup_{m \in \mathbb{N}} |Ae_m| = \sup_{m \in \infty} m = \infty.
\]

This is not a contradiction to uniform boundedness, because \( c_{00} \) is not complete. If we replace \( c_{00} \) with \( c_0 \), then however with
\[
x = (n^{-1/2})_{n \in \mathbb{N}} \in c_0,
\]

we would have
\[
(Ax)_n = n^{1/2} \to \infty.
\]

so
\[
Ax \notin c_0,
\]

that means we lose pointwise convention.

### 3.3 Open mapping theorem

We recall that if \( X, Y \) are Hausdorff, \( X \) is compact and \( f : X \to Y \) onto continuous, then \( f \) is open map. Now we define an analogous result for maps between \( F \)-space.

**3.3.18 Theorem.** Let \( A \) be a continuous map from \( F \)-space \( X \) to a TVS space \( Y \) which is continuous, linear and \( A(X) \) is of second category in \( Y \), then \( A(X) = Y \), \( A \) is open and \( Y \) is an \( F \)-space.
To prove that $A$ is open, we only need to show if it is open at $0$, i.e an open neighborhood of $0 \in X$ is mapped to a open neighborhood of $0 \in Y$. After proving this, we have for any balanced $V \in \mathcal{U}^X$, there is a balanced $W \in \mathcal{U}^Y$ such that
\[ W \subset A(V) \subset A(X). \]

but by linearity, for any $n \in \mathbb{N}$,
\[ nW \subset nA(V) \subset A(X). \]

Next, we want to show $A(V_2)$ has non-empty interior to get $W \in \mathcal{U}^Y$ with $W \subset A(V_1)$.

Next, $A(X) = A(\bigcup_{k=1}^{\infty} kV_2) = \bigcup_{k=1}^{\infty} (kA(V_2))$.

So one of $kA(V_2)$ is of second category, but scaling with $M_k$ is a homeomorphism, so $A(V_2)$ is of second category, so $A(V_2)$.

We still need to show $A(V_1) \subset A(V)$:

Repeating the nested argument with $V_n$ instead of $V_1$, we have that $A(V_{n+1})$ has a non-empty interior, so if $y_1 \in A(V_1)$, then given $y_n \in A(V_n)$, we see
\[ (y_n - A(V_{n+1})) \cap A(V_n) \neq \emptyset, \]
and
\[ (y_n - A(V_{n+1})) \cap A(V_n) \neq \emptyset. \]

Thus, there exists $x_n \in V_n$ with
\[ Ax_n \in y_n - A(V_{n+1}). \]

We can choose
\[ y_{n+1} = y_n - Ax_n, \]
so
\[ y_{n+1} \in A(V_{n+1}). \]

From $d(x_n, 0) < 2^{-n}r$, partial sums $\sum_{j=1}^{n} x_j$ form a Cauchy sequence:
\[ d(\sum_{j=1}^{n} x_j, \sum_{j=1}^{m} x_j) = d(\sum_{j=n}^{m} x_j, 0) < \sum_{j=n}^{m} 2^{-j}r \to 0, \]
which converges by completeness to some $x \in X$.

Next,
\[ \sum_{n=1}^{m} Ax_n = \sum_{n=1}^{m} (y_n - y_{n+1}) = y_1 - y_{m+1} \to y_1. \]

So we have
\[ y_1 = Ax \in A(V). \]

Finally we want to show that $Y$ is a $F$-space.