Functional Analysis, Math 7320
Lecture Notes from November 3, 2016

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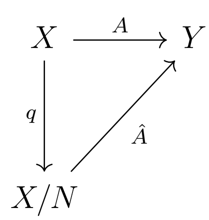
We study the open mapping theorem:

3.3.2 Theorem (Open Mapping Theorem)**.** *Let A be a map from F-space X to a topological vector space Y which is continuous, linear and $A(X)$ is second category of Y. Then, $A(X) = Y$, A is an open map, and Y is an F-space.*

Last lecture, we proved that A is an open map. As a consequence, we obtain $A(X) = Y$ since the only open subspace of Y is Y itself. We still need to show that Y is F-space.

Proof (continue). If A is one to one, since A is open, A^{-1} is well defined and continuous. Thus, A is homeomorphism from X to Y. Thus, Y is also F-space as same as X. Now, we consider in the general case. From A being continuous linear map, $N = A^{-1}(\{0\})$ is a closed subspace of X. Consider the quotient space $X/N = \{x + N : x \in X\}$ with the quotient map $q : X \to X/N$. Then q is continuous and open. Define $\tilde{A} : X/N \to Y$ by $\tilde{A}(x + N) = A(x)$ (see the diagram below). Then $\tilde{A}$ is one to one and onto. Moreover, $A = \tilde{A} \circ q$. For an open set U in Y, $\tilde{A}^{-1}(U) = \{x + N : A(x) \in U\} = \{x + N : x \in A^{-1}(U)\} = q(A^{-1}(U))$ which is open in X/N by the continuity of A and openness of the qutient map. Thus, $\tilde{A}$ is a continuous bijection. Next, if E is open in X/N, by continuity of q , $q^{-1}(E)$ is open. Then, $\tilde{A}(E) = \tilde{A}(q(q^{-1}(E))) = A(q^{-1}(E))$ is open since A is open. Thus, $\tilde{A}$ is homeomorphism. Since q preserves completeness and invariant property under translation of the metric, the quotient space X/N is also an F-space. By $\tilde{A}$ being a homeomorphism, Y is an F-space.



□

3.3.3 Remark. In the previous proof, we used the properties of fact that if N is a closed subspace of an F-space X, then the quotient space X/N is also an F-space. We provide more details about quotient space.

For a topological vector space X and a closed subspace of N of X, recall that $[x] = \{x + y : y \in N\}$ is the coset of N containing x. Then,

$$X/N = \{[x] : x \in X\}$$

with the operation $[x] + [y] = [x + y]$ and $\alpha[x] = [\alpha x]$ for $[x], [y] \in X/N$ and $\alpha \in \mathbb{R}$. This will define the quotient vector space. Let $q : X \to X$ defined by $q(x) = [x]$. We call q the quotient map. Then define the topology τ_N on X/N as follows:

$$\tau_N = \{U \subseteq X/N : q^{-1}(U) \text{ is open in } X\}.$$

Then, τ_N will be a topology on X/N and makes X/N . The following are interesting facts about the quotient space (Check [2], p31).

- τ_N makes X/N a topological vector space under the addition and multiplication as defined above.
- The quotient map $q : X \to X/N$ is linear, continuous, and open.
- If $\mathfrak{B}$ is a local base of X, then $\mathfrak{B}_N = \{p(V) : V \in \mathfrak{B}\}$ is a local base of X/N.
- Local convexity, local boundedness, metrizability, and normability properties of X will be inherited to X/N.
- X/N will be an F-space, or a Frechet space or a Banach space if X is.

Moreover, if d is an invariant metric on X. Define

$$\rho([x], [y]) = \inf\{d(x - y, z) : z \in N\}.$$

Then, we obtain

- ρ is well defined, i.e, it is not depends on the choices of x, y,
- ρ is an invariant metric, and
- The topology generated by ρ is τ_N.

Now, we are going to use the facts we have listed above to show that if X is an F space, then so is X/N.

Proof. Defining ρ as above, we have ρ is an invariant metric. We still need to show that ρ is also complete. Let u_n be a Cauchy sequence in X/N. For $\varepsilon = 1/2^k$ and apply the definition of a Cauchy sequence, we can inductively construct a subsequence u_{n_k} such that $\rho(u_{n_k}, u_{n_{k+1}}) < 1/2^k$. We will inductively choose $x_k \in u_k$ so that $d(x_k, x_{k+1}) < 1/2^k$. First, choose arbitrary $x_1 \in u_{n_1}$. After we have chosen $x_k \in u_k$, we will choose x_{k+1}. We can write $u_{n_k} = [x_k]$ and $u_{n_{k+1}} = [y]$ for some $y \in u_{n_{k+1}}$. Since $\rho(u_{n_k}, u_{n_{k+1}}) = \inf\{d(x_k - y, z) : z \in N\} < 1/2^k$, there is $z \in N$ such that $d(x_k - y, z) < 1/2^k$. By the invariant property, $d(x_k - y, z) = d(x_k, y + z) < 1/2^k$. Then, we choose $x_{k+1} = y + z \in [y] = u_{n_{k+1}}$. By this construction, for $k < l$, we have $d(x_k, x_l) < d(x_k, x_{k+1}) + ... + d(x_{l-1}, x_l) < \sum_{j=k}^{l-1} 2^j < 2/2^k$. This implies that x_k is a Cauchy sequence in X and thus it converges to some $x \in X$. Thus, u_{n_k} converges to $q(x)$ by the continuity of the quotient map q. Since a subsequence of a Cauchy sequence converges, it forces the whole sequence converges to the same limit. This proved that ρ is a complete metric. □

The open mapping theorem can be applied to specific cases. First, we consider the case when A is linear continuous bijection and both X and Y are F-spaces.

3.3.4 Corollary. *If $A : X \to Y$ is a bijection continuous linear map between F-spaces X and Y, then A is homeomorphism.*

Proof. From the statement, we have (i) X is an F-space X, (ii) A is continuous and linear from X to a topological vector space Y, and (iii) $A(X) = Y$ is an F-space and thus of the second category in themselves by Baire's theorem. As a consequence, the open mapping theorem concludes A is an open map. Since A is bijective, A^{-1} is well defined. Since A is open, A^{-1} is continuous. In conclusion, A is continuous and has a continuous inverse. Therefore, A is homeomorphism. □

3.3.5 Remark. (i) From the proof of the previous corollary, we also have any continuous linear map between F-spaces is always open.
(ii) If (X, τ_1) and (X, τ_2) are topological vector spaces which are both F-spaces. If τ_1 and τ_2 are comparable, i.e, one is finer than the other, then both are equal. To prove this, with out loss of generality, we assume that $\tau_2 \subset \tau_1$. Then, the identity map $Id : (X, \tau_1) \to (X, \tau_2)$ is linear and continuous. By the corollary, Id is homeomorphism. Thus, $\tau_1 = \tau_2$. This suggests that any F-space can not be compared to another. On the other hand, if (X, τ_1) is an F-space and $\tau_1 \subsetneq \tau_2$ (or $\tau_2 \subsetneq \tau_1$), (X, τ_2) can not be F-space.

More specifically, we consider consequences of the open mapping theorem on Banach spaces.

3.3.6 Remark. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)$ be normed spaces. If A is a continuous linear map from X to Y, there exists $M > 0$ such that $\|Ax\|_Y \leq M\|x\|_X$ for all $x \in X$.

3.3.7 Corollary. *If X and Y are Banach spaces and $A : X \to Y$ is continuous linear bijection, then there exist $m, M > 0$ with*

$$m\|x\|_X \leq \|Ax\| \leq M\|x\|_X.$$

Proof. Since A is linear and continuous, the imidiate consequence is that there is $M > 0$ such that

$$\|Ax\|_Y \leq M\|x\|_X$$

for all $x \in X$. Since X is a Banach space, it is complete metric space and so F-space. Similarly, since A is onto, $A(X) = Y$ which is a Banach space. Thus, $A(X)$ is F-space which also is of the second category. Therefore, applying the open mapping theorem to the continuous linear map A, we obtain A is an open map. Since A is one to one, A^{-1} is well defined and thus continuous by the openess of A. Therefore, there is $\tilde{M} > 0$ such that

$$\|A^{-1}y\|_X \leq \tilde{M}\|y\|_Y$$

for all $y \in Y$. Let $x \in X$. Since A is bijective, there is a unique $y \in Y$ such that $y = Ax$ and $x = A^{-1}y = A^{-1}Ax$. Replacing y by Ax into the above inequality, we obtain

$$\|x\|_X \leq \tilde{M}\|Ax\|_Y.$$

By choosing $m = 1/\tilde{M}$, we conclude that

$$m\|x\|_X \leq \|Ax\|_Y \leq M\|x\|_X$$

as we desire. □

3.3.8 Remark. (i) The previous theorem means norms on X and Y are equivalent.
(ii) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X which both are complete norms generating the same topology. Let $Id : X \to X$ be the identity map. Then Id is a continuous linear bijection. By replacing $Ax = Ix = x$ to the previous corollary, we have that

$$m\|x\|_2 \leq \|x\|_1 \leq M\|x\|_2.$$

On the other words, all norms in a Banach space X are equivalent.

3.4 Warm Up: Internal vs Interior

3.4.9 Definition. Let V be a vector space, $S \subseteq V$. A point $x \in S$ is called an internal point if for every $y \in V$, there is $\varepsilon > 0$ such that $\{x + ry : |r| < \varepsilon\} \subseteq S$.

3.4.10 Remark. A set $S \subseteq V$ for which 0 is an internal point is absorbing because for all $y \in V$, there is $n \in \mathbb{N}$ such that $y/n \in S$, i.e., $V = \bigcup_{n=1}^{\infty} nS$.

3.4.11 Remark. If 0 is an interior point of a subset S of a topological vector space X, then it has a balanced open subset U such that $0 \in U \subseteq S$. In addition, $X = \bigcup_{n=1}^{\infty} nU$. Let $x \in X$. Hence, $x \in nU$ for some $n \in \mathbb{N}$. Thus, $\frac{1}{n}x \in U$. Since U is balanced, $\frac{\alpha}{n}x \in U$ for all $|\alpha| < 1$. Therefore, an interior point is also an internal point. But, the converse is not true as the following examples.

3.4.12 Example. We provide some examples on $\mathbb{R}^2$.

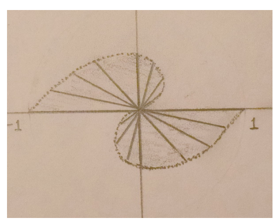


Image of S_1

1. In $\mathbb{R}^2$, for $\theta \in (0, \pi)$, let $A_\theta = (-\theta/\pi, \theta/\pi)(\cos\theta, \sin\theta)$. Define $S_1 = (\bigcup_{\theta \in (0,\pi)} A_\theta) \cup \{(x, 0) : x \in (-1, 1)\}$. Then, 0 is an internal point of S_1 but not interior point.

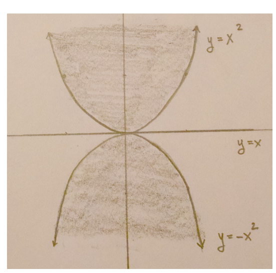


Image of S_1

2. Let $S_2 = \{(x, y) : y > x^2\} \cup \{(x, 0) : x \in R\} \cup \{(x, y) : y < -x^2\}$. Then 0 is an internal point but not interior point of S_2.

3.4.13 Remark. In a finite dimensional topological vector space, interior points and internal points in a convex set coincide (check [2]). Since a real or complex topological vector space of dimension n is homeomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$. It suffices to prove the statement in $\mathbb{R}^n$ and $\mathbb{C}^n$. To get a perspective about this statement, we consider in the case of $\mathbb{R}^2$. Let 0 be an internal point of a convex set $S \subseteq \mathbb{R}^2$. Choose $v_1 = (1,1)$ and $v_2 = (1,-1)$. Note that these two vector form a basis of $\mathbb{R}^2$. Since 0 is an internal point of S, there are positive real numbers r_1 and r_2 such that $\{\alpha v_1 : |\alpha| < r_1\} \subseteq S$ and $\{\beta v_2 : |\beta| < r_2\} \subseteq S$. Choose $r = \min\{r_1, r_2\}/2$. Then $rv_1 = (r,r), -rv_1 = (-r,-r), rv_2 = (r,-r), -rv_2 = (-r,r) \in S$ since $|\pm r| < r_1$ and $|\pm r| < r_2$. For any $0 \le t \le 1$, $t(r,r) + (1-t)(-r,r) = ((2t-1)r, r) \in S$ and $t(r,-r) + (1-t)(-r,-r) = ((2t-1)r, -r) \in S$ by the convexity. Again, by convexity, for any $0 \le \tilde{t} \le 0$, $\tilde{t}((2t-1)r, r) + (1-\tilde{t})((2t-1)r, -r) = ((2t-1)r, (2\tilde{t}-1)r) \in S$. For $(x,y) \in \mathbb{R}^2$ where $|x| < r$ and $y < |r|$, choose $t = (x/r+1)/2$ and $\tilde{t} = (y/r+1)/2$ which both are in $[0,1]$. Thus, $(x,y) = ((2t-1)r, (2\tilde{t}-1)r)$. Therefore, $(0,0) \in \{(x,y) : |x| < r, |y| < r\} \subseteq S$ which is open. Therefore, 0 is an internal point. We can extend this idea to prove the statement for $\mathbb{R}^n$ and $\mathbb{C}^n$. For an internal point of S which is not 0, we can replace S by $S - x$.

3.5 The Closed Graph Theorem

3.5.14 Definition. Let $f : X \to Y$. The graph of f is

$$\Gamma(f) = \{(x, f(x) : x \in X\} \subset X \times Y.$$

3.5.15 Theorem. *Let X is a topological space and Y is a Hausdorff space. If $f : X \to Y$ is continuous, then the graph of f, $\Gamma(f)$ is closed in the product topology.*

Proof. Take $\Omega = X \times Y \setminus \Gamma(f)$ and $(x,y) \in \Omega$. Thus, $y \neq f(x)$. By Hausdorff property of Y, there exist disjoint open sets V, W such that $f(x) \in V$ and $y \in W$. Since f is continuous, $f^{-1}(V)$ is open, and hence, $f^{-1}(V) \times W$ is an open set in the product topology $X \times Y$ containing (x,y). Moreover, if $(a,b) \in f^{-1}(V) \times W$, $f(a) \in V$ and $b \in W$. Since V and W are disjoint, $f(a) \neq b$, i.e., $f^{-1}(V) \times W \subseteq \Omega$. Therefore, Ω is open; hence, $\Gamma(f) = X \times Y \setminus \Omega$ is closed. □

Under certain assumptions, we can show the converse of this theorem.

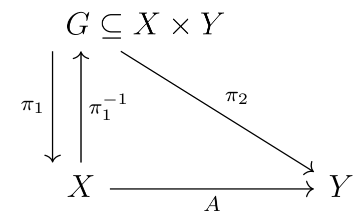
3.5.16 Theorem. *Let $A : X \to Y$ be a linear map between F-spaces X and Y. Then A is continuous if and only if the graph of f, $\Gamma(f)$ is closed.*

Proof. We observe that if d_X and d_Y are invariant metrices on X and Y respectively, then so is

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Also, we note that compactness is preserved, so $X \times Y$ is an F-space. By linearity, $\Gamma(f)$ is a subspace of $X \times Y$. Assume that A is continuous, then $\Gamma(f)$ is closed, thus it is complete. Hence, $\Gamma(f)$ is an F-space.

Assume that $\Gamma(f)$ is closed, that is F-space. Let $\Pi_1 : \Gamma(f) \to X$ be a projection map on X and $\Pi_2 : X \times Y \to Y$ be a projection map on Y. We see that Π_1 is a bijective continuous map. Thus, the composition $\Pi_2 \circ \Pi_1^{-1} = A$ is continuous. □



3.5.17 Remark. To show that the graph of f is closed, we can verify that if $x_n \to x$, then $f(x_n) \to f(x)$.

4 Convexity

Next, we study spaces through their duals.

4.1 Hahn Banach Theorem

For a topological vector space X, a real (or complex) **functional** on X is a function $f : X \to \mathbb{R}$ (or $\mathbb{C}$), that is f maps an element in X to a real number (or a complex number). Assume that we have a linear functional f defined on a subspace of the whole space. Under some constrains of f, we might be able to extend f to a functional on X. The Hanh Banach Theorem shows that a function on a subspace can be extended to a functional on the whole space if it is dominated by a nice functional on the whole space.

4.1.1 Theorem (Hahn Banach Theorem on $\mathbb{R}$)**.** *Let V be a real vector space and $p : V \to \mathbb{R}$ satisfying*

- $p(x + y) \leq p(x) + p(y)$.
- $p(\alpha x) = \alpha p(x)$.

Let $Y \subseteq V$ be a linear subspace and F a linear functional $f : X \to \mathbb{R}$ such that $f \leq p|_Y$. Then, there is a linear functional $F : V \to \mathbb{R}$ such that $F_Y = f$ and also $F \leq p$.

References

[1] Rudin, Walter., *Functional Analysis, 2nd*, McGraw Hill Education, 1973

[2] Kantorovitz, Shmuel., *Introduction to Modern Analysis*, Oxford University Press, 2003, p. 134.