# Functional Analysis, Math 7320
# Lecture Notes from November 3, 2016

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# 3 Completeness

## 3.1 Open Mapping Theorem

**3.1.14 Theorem.** *Open Mapping Theorem*

*Let X be an F-space, Y be a topological vector space. Let $A : X \to Y$ be a continuous, linear map, and $A(X)$ is of 2nd-category of Y. Then, $A(X) = Y$, A is open, and Y is an F-space.*

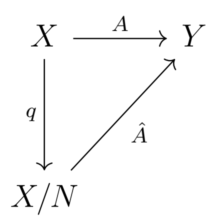
*Proof.* (cont'd)

Last time we showed that $A$ is an open map, and $A(X) = Y$. We still need to show Y is an F-space. Notice that if $A$ is one-to-one, then $A$ is a homeomorphism. This is because $A(X) = Y$ implies A is a bijective, and A being a continuous, open map implies that its inverse $A^-1$ is continuous.

However, in general, A is not always one-to-one. To get around this, we will construct a 1-1 function between Y and an F-space we already know.

First, define a quotient map $q : X \to X/N$,where $N = A^{-1}(\{0\})$. Note that q is linear, and onto, and the kernel $N$ is a closed subsace of X.

Define $\tilde{A} : X/N \to Y$, $\tilde{A}(x + N) = Ax$. Then $\tilde{A}$ is a bijection, and $A = \tilde{A} \circ q$




To show that $\tilde{A}$ is open, take a set $E$ open (w.r.t final topology) in $X/N$. By continuity of quotient map $q$, $q^{-1}(E)$ is open.

$\Longrightarrow$ $A(q^{-1}(E))$ is open because $A$ is open (as shown earlier)

$\Longrightarrow$ $\tilde{A}$ is open, continuous and 1-1

$\Longrightarrow$ $\tilde{A}$ is homeomorphism

What's left to show is $X/N$ is an F-space.

For translation-invariance of $X/N$, let $d$ be the translation-invariant metric on X, and define a mtric $\rho$ on $X/N$ by:

$$\rho(q(x), q(y)) = inf\{d(x-y, z) : z \in N\}$$

Then $\rho$ is the invariant metric on the quotient space $X/N$.

For completeness of $X/N$, let $\{u_n\}_n$ be a Cauchy sequence in $X/N$ (with respect to the metric $\rho$), then there exists a subsequence $\{u_{n_i}\}_i$ such that $\rho(u_{n_i}, u_{n_{i+1}}) < 2^{-i}$. Since $q$ is an onto map, we can select $x_i$ such that $q(x_i) = u_{n_i}$, and $d(x_i, x_{i+1}) < 2^{-i}$. Then, $x_i$ is a Cauchy sequence, hence, by the completeness of metric $d$, $x_i$ converges to some element $x \in X$. Since $q$ is continuous, $u_{n_i} = q(x_i)$ converges to $q(x) \in X/N$. The Cauchy sequence $u_n$ has a convergent subsequence $u_{n_i}$, so $u_n$ also coverges. Hence, $X/N$ is complete in the metric $\rho$. □

**3.1.15 Corollary.** *Each bijective, continuous, linear map between F-spaces is a homeomorphim*

*Proof.* Let $X, Y$ be F-spaces, and $f : X \to Y$ be a bijective, continuous, linear map. Since $f$ is CTS and linear, by Open Mapping Theorem, $f$ is an open map. Hence, $f^{-1}$ is continuous. We conclude that $f$ is a homeomorphism. □

*3.1.16 Remark.* In the corollary above, the inverse $f^{-1}$ is also bounded.

**3.1.17 Corollary.** *Let $X$, $Y$ be Banach spaces, and $A : X \to Y$ be a continuous, linear bijection. Then, there exists constants $M$, $m > 0$ such that for all $x \in X$:*

$$m\|x\|_X \leq \|Ax\|_Y \leq M\|x\|_X$$

*Proof.* First, $A$ is continuous and linear, so the map $A$ is bounded. Therefore, by definition of boundedness of an operator norm, there exists a constant $M > 0$ such that for all $x \in X$:

$$\|Ax\|_Y \leq M\|x\|_X$$

Similarly, $A$ is bijective, continuous, and linear, so by the preceding corollary, $A^{-1}$ is continuous; hence, bounded. Therefore, there exists a constant $\tilde{m} > 0$ such that for any $y \in Y$

$$\|A^{-1}y\|_X \leq \tilde{m}\|y\|_Y$$

Since $A$ is a bijective, for each $y \in Y$, there is a corresponding $x \in X$ such that $Ax = y$. Hence,

$$\|x\|_X \leq \tilde{m}\|Ax\|_Y$$

Equivalently, $\frac{1}{\tilde{m}}\|x\|_X \leq \|Ax\|_Y$. Choose $m = 1/\tilde{m}$, we have the desired result. □

*3.1.18 Remarks.*

1. The choice of $m$, and $M$ is independent of $x \in X$.
2. The norm on Y is equivalent to the norm on X.

## 3.2 Internal vs Interior

**3.2.19 Definition.** Let V be a vector space, and $S \subset V$. A point $x \in S$ is called an internal point if for each $y \in V$, there is $\epsilon > 0$ s.t. $x + (-\epsilon, \epsilon)y \subset S$.

*3.2.20 Remark.* A set $S \subset V$ for which $0$ is an internal point is absorbing because for all $y \in V$, there is $n \in \mathbb{N}$ s.t. $\frac{y}{n} \in S$, i.e. $V = \bigcup_{n=1}^{\infty}(nS)$.

In general, an internal point is not an interior point. For example, let $S = \{(x, y) \subset \mathbb{R}^2 : y \geq x^2\} \cup \{(x, y) \subset \mathbb{R}^2 : y \leq -x^2\} \cup \{(x, 0) \subset \mathbb{R}^2 : x \in R\}$. Then the origin is an internal point, but not an interior point of $S$

## 3.3 Closed Graph Theorem

**3.3.21 Definition.** Given sets $X$, $Y$, and a function $f : X \to Y$, then $\Gamma(f) = \{(x, f(x)\}_{x \in X} \subset X \times Y$ is called the graph of $f$

**3.3.22 Theorem.** *Closed Graph Theorem If $X$ is a topological space, $Y$ is a Hausdorff space, and $f : X \to Y$ is continuous, then $\Gamma(f)$ is closed in the product topology.*

*Proof.* Let $\Omega = X \times Y \setminus \Gamma(f)$, and take $(x_0, y_0) \in \Omega$. Then $y_0 \neq f(x_0)$

Since Y is Hausdorff, there exist open sets $V$ containing $y_0$, and $W$ containing $f(x_0)$ such that $V \cap W = \emptyset$
$\implies V \times W$ is open in $Y \times Y$ (w.r.t product topology)
Since $f$ is continuous, $f^{-1}(W)$ is open in X, hence $f^{-1}(W) \times V$ is open in $X \times Y$. Moreover, for any $(x, y) \in f^{-1}(W) \times V$, we have $f(x) \in W$, and $y \in V$, but $V$ and $W$ are disjoint, so $f(x) \neq y$. This implies $f^{-1}(V) \times W \cap \Gamma(f) = \emptyset$. Hence, $f^{-1}(W) \times W \subset \Omega$ is an open set, and contains $(x_0, y_0)$. We conclude that $\Omega$ is open, i.e. $\Gamma(f)$ is closed.

□

Under some circumstances, the converse is also true

**3.3.23 Theorem.** *Let $A : X \to Y$ be a linear map between F-spaces. Then,*
*$A$ is continuous $\iff$ $\Gamma(A)$ is closed in $X \times Y$*

*Proof.* First, observe that the metrices $d_X$ and $d_Y$ are invariant on $X$, $Y$ resp, and so is $d$, where $d$ is the metric on $X \times Y$, defined by:

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

Both $X$ and $Y$ are complete, so $X \times Y$ is complete. Hence, $X \times Y$ is an F-space. Next, we'll show $\Gamma(A)$ is a subspace of $X \times Y$. For any $(x_1, A(x_1))$ and $(x_2, A(x_2)) \in \Gamma(A)$, $c_1, c_2 \in \mathbb{R}$, by linearity of A, we have :

$$c_1(x_1, A(x_1)) + c_2(x_2, A(x_2)) = (c_1x_1 + c_2x_2, c_1A(x_1) + c_2A(x_2)) \quad (1)$$
$$= (c_1x_1 + c_2x_2, A(c_1x_1 + c_2x_2)) \quad (2)$$

Therefore, $\Gamma(A)$ is a subspace of $X \times Y$. For the forward direction, assume A is continuous. Then, $\Gamma(A)$ is closed in $X \times Y$ (by Closed graph theorem). $X \times Y$ is complete, so $\Gamma(A)$ is complete. Hence, $\Gamma(A)$ is an F-space. Conversely, assume $\Gamma(A)$ is closed. Then, $\Gamma(A)$ is an F-space (by the same argument above). Define the projection maps:

$$\pi_1 : \Gamma(A) \to X$$

$$(x, Ax) \mapsto x$$

and

$$\pi_2 : X \times Y \to Y$$

$$(x, y) \mapsto y$$

Projection maps $\pi_1$ is continuous (with $X \times Y$ endowed with product topology), 1-1, and onto. By open mapping theorem, $\pi_1$ has a bounded inverse $\pi_1^{-1}$. Hence, $\pi_1^{-1}$ is continuous. Therefore, the composition $\pi_2 \circ \pi_1^{-1} = A$ is continuous □

# 4 CONVEXITY

In this section, we'll study spaces through their duals

**4.0.1 Theorem.** *Hahn-Banach) Let $V$ be a real vector space, and $p$ be a function on $V$ satisfying:*

1. *(sublinearity) $p(x+y) \neq p(x) + p(y)$, for all $x, y \in V$*
2. *(homogeneity) $p(\alpha x) = \alpha p(x)$, for $\alpha > 0$*

*Let $Y \subset V$ be a linear subspace, $f$ be a linear functional $f : Y \to R$ s.t $f \leq p|_Y$ Then, there is a linear functional $F$ on $V$ with $F|_Y = f$, and $F \leq p$*

*4.0.2 Remarks.* 1. In the above theorem, $p$ doesn't have to be a seminorm. For example, we can take $p(x) = max(0, x), x \in \mathbb{R}$, then $p$ satisfies sublinearity and homogeneity while $p$ is not a seminorm.

2. No Banach space is needed