# Functional Analysis, Math 7320
# Lecture Notes from November 08, 2016

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**Warm-up**: *Internal vs. convexity*
We revisit the sequence space from last time. Let,

$$C_{0+} := \{x \in C_{00} \text{ with last (if existing) non-zero entry strictly positive}\}.$$

*1.3.20 Question.* Does $C_{0+}$ have an internal point?

*1.3.21 Answer.* **No**. We show that no point $x \in C_{0+}$ is internal. Assume that $x_n > 0$ is the last non-zero entry of an arbitrary point $x \in C_{0+}$. Then, pick $y \in C_{0+}$ with last non-zero entry $y_{n+1} > 0$. We observe that for any $\epsilon > 0$, the point $x - \epsilon y$ is *not* in $C_{0+}$. So $x$ is not internal.

*1.3.22 Question.* Is $C_{0+}$ convex?

*1.3.23 Answer.* **Yes**. Let $x, y \in C_{0+}$ be chosen arbitrarily. Let $z = \epsilon x + (1-\epsilon)y \in C_{0+}$ be any point on the line segment joining $x$ and $y$, with $\epsilon \in (0,1)$. Call $m, n \in \mathbb{N}$ the indices of the last non-zero entries of $x$ and $y$ respectively, and $N := \max(m,n)$. Then, $z_N = \epsilon x_N + (1-\epsilon)y_N > 0$ is the last non-zero entry of $z$, which make it a point of $C_{0+}$.

*1.3.24 Remark.* A line segment in $\mathbb{R}^2$ is another example of a convex set that has no internal point.

**1.3.25 Theorem** (Hahn-Banach over $\mathbb{R}$)**.** *Let $V$ be a real vector space and $p$ a functional on $V$ satisfying the following:*

1. *$p(x+y) \leq p(x) + p(y)$ for all $x, y \in V$, (subadditivity)*
2. *$p(\alpha x) = \alpha p(x)$ for all $x \in V$ and $\alpha \geq 0$. (positive homogeneity)*

*Let $Y \subseteq V$ be a linear subspace and $f : Y \to \mathbb{R}$, a linear functional dominated by $p$, i.e. such that $f \leq p_{|_Y}$. Then, there is a linear functional $\mathcal{F}$ on $V$ which extends $f$, $\mathcal{F}_{|_Y} = f$, and preserves the inequality: $\mathcal{F} \leq p$.*

*Proof.* We start by an application of Zorn's lemma. Consider the collection of linear extensions of $f$ dominated by $p$, $\Gamma := \{(g, V_g)\}$, with each $V_g$ being a linear subspace of $V$ containing $Y$, and each $g : V_g \to \mathbb{R}$ a linear functional satisfying $g_{|_Y} = f$ and $g \leq p_{|_{V_g}}$. We define a partial order on $\Gamma$ as follows: $(g_1, V_{g_1}) \preceq (g_2, V_{g_2})$ if and only if,

- $V_{g_1} \subseteq V_{g_2}$, AND

- $g_2\big|_{V_{g_1}} = g_1$.

Recall that linearly ordered subsets of $\Gamma$ are called *chains*. Let $\{(g, V_g)\}_{g\in G}$ be a chain and set $D := \bigcup_{g\in G} V_g$. Then, the function $F$ defined on $D$ by $F(x) := g(x)$ if $x \in V_g$ for some $g \in G$, is well-defined, linear, and satisfies $F \le p\big|_D$. Thus, $F \in \Gamma$ is an upper bound for the chain. By Zorn's lemma, there exists a maximal element $(\mathcal{F}, V_{\mathcal{F}})$ of $\Gamma$.

We now set to show that $V_{\mathcal{F}} = V$, which will complete the proof. Assume for *contradiction* that $V_{\mathcal{F}} \subsetneq V$. Then we may choose $x_0 \in V \setminus V_{\mathcal{F}}$ and set $V_1 := \text{span}\{V_{\mathcal{F}}, x_0\}$. Each vector $x \in V_1$ can then be uniquely decomposed as $x = y + \alpha x_0$ with $(y, \alpha) \in V_{\mathcal{F}} \times \mathbb{R}$. Next, define a linear functional $f_1 : V_1 \to \mathbb{R}$ as follows: $f_1(y) := \mathcal{F}(y)$ if $y \in V_{\mathcal{F}}$ and $f_1(x_0) := \beta$, for an arbitrary $\beta \in \mathbb{R}$. By construction we have that for all $x = y + \alpha x_0 \in V_1$, $f_1(x) = \mathcal{F}(y) + \alpha\beta$. We will now choose $\beta$ (which was arbitrary up until now) in such a way that the inequality,

$$f_1 \le p\big|_{V_1}, \tag{1}$$

becomes true. Note that if we succeed, then the mere existence of $(f_1, V_1)$ will violate the maximality of $(\mathcal{F}, V_{\mathcal{F}})$, which will generate our contradiction. Inequality (1) is equivalent to, $f_1(y) + \alpha\beta \le p(y + \alpha x_0)$, holding true for any $(y, \alpha) \in V_{\mathcal{F}} \times \mathbb{R}$. In particular, for $y, y' \in V_{\mathcal{F}}$ and $\alpha = \pm 1$, the following equations must hold:

$$f_1(y) + \beta \le p(y + x_0), \tag{2}$$
$$f_1(y') - \beta \le p(y' - x_0). \tag{3}$$

We combine them to get the equivalent systems:

$$\begin{aligned} & -\beta + f_1(y') - p(y' - x_0) \;\le\; 0 \;\le\; p(y + x_0) - f_1(y) - \beta \\ \iff\quad & f_1(y') - p(y' - x_0) \;\le\; \beta \;\le\; p(y + x_0) - f_1(y). \end{aligned} \tag{4}$$

Now, we explain why the inequality, $f_1(y') - p(y' - x_0) \le p(y + x_0) - f_1(y)$, is true for all $y, y' \in V_{\mathcal{F}}$, and we derive a direct consequence of it.

$$\begin{aligned} f_1(y') + f_1(y) &= f_1(y + y') \\ &= f_1(y + x_0 + y' - x_0) \\ &= \mathcal{F}(y + x_0 + y' - x_0) && \text{(by construction } y + y' \in V_{\mathcal{F}}) \\ &\le p(y + x_0 + y' - x_0) \\ &\le p(y + x_0) + p(y' - x_0) && \text{(by subadditivity of } p \text{ on } V) \\ \implies \quad f_1(y') - p(y' - x_0) &\le p(y + x_0) - f_1(y) \end{aligned}$$

$$\implies \sup_{y'\in V_{\mathcal{F}}} \left(f_1(y') - p(y' - x_0)\right) \;\le\; \inf_{y\in V_{\mathcal{F}}} \left(p(y + x_0) - f_1(y)\right).$$

We see that, as long as both sides of the last inequality are not both equal to $+\infty$ or $-\infty$, there exists a $\beta \in \mathbb{R}$ satisfying our inequality (4). But this potential obstacle cannot happen since,

$$-\infty < \sup_{y'\in V_{\mathcal{F}}} \left(f_1(y') - p(y' - x_0)\right) \le \inf_{y\in V_{\mathcal{F}}} \left(p(y + x_0) - f_1(y)\right) < \infty,$$

as none of the sets of which we are taking the $\sup$ or $\inf$ are empty. With $\beta$ such chosen, we are now able to prove that (1) is true for any $\alpha \in \mathbb{R}$. We proceed by cases.

If $\alpha > 0$,

$$f_1(y + \alpha x_0) = \alpha f_1\left(\frac{y}{\alpha} + x_0\right) \leq \alpha p\left(\frac{y}{\alpha} + x_0\right) = p(y + \alpha x_0). \qquad \text{(where we used (2))}$$

If $\alpha = 0$, $\quad f_1(y) = \mathcal{F}(y) \leq p(y)$.

If $\alpha < 0$,

$$f_1(y+\alpha x_0) = |\alpha| f_1\left(\frac{y}{|\alpha|} - x_0\right) \leq |\alpha| p\left(\frac{y}{|\alpha|} - x_0\right) = p(y+\alpha x_0). \qquad \text{(where we used (3))}$$

□

When $V$ is a $\mathbb{C}$-vector space, the real part of a linear functional $f : V \to \mathbb{C}$ determines its imaginary part, by linearity: $f(x) = \mathbf{Re} f(x) - i\mathbf{Re} f(ix)$, for all $x \in V$. We can therefore apply a similar strategy as above, to prove the following theorem.

**1.3.26 Theorem** (Hahn-Banach over $\mathbb{C}$)**.** *Let $X$ be a complex vector space and $p$ a seminorm on $X$. Let further, $Y \subseteq X$ be a linear subspace and $f : Y \to \mathbb{C}$, a linear functional satisfying $|f| \leq p_{|Y}$. Then, there is a linear functional $F : X \to \mathbb{C}$ which extends $f$, $F_{|Y} = f$, and preserves the inequality: $|F| \leq p$.*

*Proof.* First, we see $X$ as a real vector space, on which the linear functional $\mathbf{Re} f(x) : Y \to \mathbb{R}$ may be extended to $g : X \to \mathbb{R}$ by virtue of theorem 1.3.25, with $g \leq p$. Next, we define $F(x) := g(x) - ig(ix)$ on $X$, which by construction, is complex-linear, and agrees with $f$ on $Y$. It only remains to show that $|F| \leq p$. Let $x \in X$ be arbitrary. Since $F(x)$ lies in the complex plane, there is a complex number $\alpha$ with norm $1$ such that $\alpha F(x) = |F(x)|$. Geometrically, we can think of this $\alpha$ as a rotation map centered at the origin and taking the point $F(x)$ to the positive-half of the real line. Since $F$ is linear, we also have $\alpha F(x) = F(\alpha x)$, and since $F(\alpha x) \in \mathbb{R}$ by construction, we must have $F(\alpha x) = \mathbf{Re} F(\alpha x) = g(\alpha x)$. Since $g$ is dominated by $p$ and $p$ is a seminorm, we get $|F(x)| = g(\alpha x) \leq p(\alpha x) = p(x)$, which completes the proof. □

*1.3.27 Remark.* The preceding theorem applies to normed spaces and shows the existence of an abundance of linear functionals.

**1.3.28 Corollary.** *If $X$ is a normed space and $x_0 \in X$, then there exists a linear functional $f$ such that $f(x_0) = ||x_0||$ and $|f(x)| \leq ||x||$ for all $x \in X$.*

*Proof.* If $x_0 = 0$, let $f \equiv 0$. Otherwise, set $p(x) := ||x||$ for all $x \in X$, $Y :=$ span$\{x_0\}$, define the linear functional $f(\alpha x_0) := \alpha ||x_0||$, and extend it to $X$ using the preceding theorems[1]. □

Next, we discuss separation.

**1.3.29 Definition.** Let $V$ be a vector space and $M, N \subseteq V$ two subsets. A linear functional $f$ on $V$ is said to *separate* $M$ and $N$ if,

$$\sup \mathrm{Re}\left[f(N)\right] \leq \inf \mathrm{Re}\left[f(M)\right],$$

where $f(N)$ and $f(M)$ denote the respective images of $N$ and $M$ under $f$.

[1] Note that an easy consequence of theorem 1.3.25 is that $-p(-x) \leq -\mathcal{F}(-x) = \mathcal{F}(x)$, for all $x \in V$, which entails $|\mathcal{F}| \leq p$ when $p$ is a seminorm or a norm.