3.3.23 Proposition. A linear functional \( f \) separates between 2 sets \( M \) and \( N \) if and only if it separates between \( M - N \) and \( \{0\} \).

Proof. Let \( V \) be a vector space over \( K \), \( M, N \subseteq V \) two subsets, and \( f : V \to K \) a linear functional. Then \( x \mapsto \text{Re } f(x) \) is a real-linear\(^1\) functional \( V \to \mathbb{R} \), when \( V \) is considered as an \( \mathbb{R} \)-vector space. Hence, for each \( (x, y) \in M \times N \),

\[
\text{Re } f(x) \leq \text{Re } f(y) \iff \text{Re } f(x - y) \leq 0 = \text{Re } f(0).
\]

The result follows by definition. \( \Box \)

Warm-up: Recall that \( C_{0+} \) is a convex subset of \( C_{00} \).

3.3.24 Claim. There is no linear functional on \( C_{00} \) that separates \( C_{0+} \) from \( \{0\} \).

Proof. Suppose for contradiction that there is a non-zero functional \( f : C_{00} \to \mathbb{R} \) such that \( f(C_{0+}) \subseteq \mathbb{R}^+ \). Call \( e_j \ (j \in \mathbb{N}) \) the canonical vector from \( C_{00} \) which has its \( j \)th entry equal to 1 and all the other ones to 0. Clearly \( e_j \in C_{0+} \) and \( f(e_j) \geq 0 \) for all \( j \in \mathbb{N} \). Furthermore, the fact that \( \{e_j\}_{j \in \mathbb{N}} \) forms a basis for \( C_{00} \) and the assumption that \( f \neq 0 \) imply that \( f(e_j) > 0 \) for some \( j \in \mathbb{N} \). If \( f(e_{j+1}) = 0 \), then \( -e_j + e_{j+1} \in C_{0+} \), and \( f(-e_j + e_{j+1}) < 0 \) generates a contradiction. If \( f(e_{j+1}) > 0 \), we define

\[
v := -e_j + \frac{f(e_j)}{2f(e_{j+1})} e_{j+1} \in C_{0+},
\]

which yields, using linearity of \( f \), \( f(v) < 0 \), and also generates a contradiction. \( \Box \)

3.3.25 Question. Is it possible to separate \( \{0\} \) from any other set in \( C_{00} \) with a linear functional?

3.3.26 Theorem (Masur). Let \( M, N \) be disjoint non-empty convex sets in a vector space \( V \). If at least one of these sets, say \( M \), has an internal point, then there exists a non-zero linear functional that separates \( M \) and \( N \).

We will only consider the real case here, that is when \( K = \mathbb{R} \). We will first present a lemma.

\(^1\)Meaning that the scalar field is \( \mathbb{R} \).

\(^2\)The fact that 0 \( \in C_{0+} \) does not impede separation by itself, as we defined separation with a large inequality.
3.3.27 Lemma. 1. A linear functional $f : V \to \mathbb{R}$ separates $M$ and $N$ if and only if it separates $M - p$ and $N - p$.

2. A point $p \in M$ is an internal point of $M$ if and only if $0$ is an internal point of $M - p$.

3. (a) For any $p \in V$, the set $M$ is convex if and only if $M - p$ is convex.

(b) If $M, N \subseteq V$ are convex, then so is $M - N$.

Proof of lemma. 1. This is a direct consequence of the following equalities,

$$(\sup f(M)) - f(p) = \sup (f(M) - f(p)) = \sup f(M - p),$$

which also hold true for $(\inf f(N)) - f(p)$.

2. This is immediate from the definition, given that for all $x \in V$ and all $\epsilon \in (-1, 1)$, $p + \epsilon x \in M$ $\iff$ $\epsilon x \in M - p$.

3. We start by proving (b). Let $m_1, m_2 \in M$, $n_1, n_2 \in N$ and $0 \leq \lambda \leq 1$. The result is immediate considering,

$$\lambda (m_1 - n_1) + (1 - \lambda)(m_2 - n_2) = \lambda m_1 + (1 - \lambda)m_1 - (\lambda n_1 + (1 - \lambda)n_2) \in M - N.$$

Now, (b) implies one implication in (a), when $N = \{p\}$. For the converse, assume that $M - p$ is convex, let $m_1 - p, m_2 - p \in M - p$ and $\lambda \in [0, 1]$. We obtain the result as follows:

$$\lambda (m_1 - p) + (1 - \lambda)(m_2 - p) \in M - p \Rightarrow \lambda m_1 + (1 - \lambda)m_2 \in M.$$

Proof of Masur’s theorem in the real case. By lemma ??, we may assume without loss of generality that $0$ is an internal point of $M$. Next, let $x_0 \in N$ and define $K := M - N + x_0$. The first part of lemma ?? together with proposition ?? imply the following chain of equivalence: A linear functional separates between $\{x_0\}$ and $K$, if and only if it separates between $\{0\}$ and $M - N$, if and only if it separates between $M$ and $N$. So our task, at this point, becomes to prove the existence of a non-zero linear functional on $V$ that separates between $K$ and $\{x_0\}$. For this, we will use the Minkowski functional $\mu_K : V \to [0, \infty)$, of $K$. Recall that,

$$\mu_K(x) := \inf \left\{ t > 0 : t^{-1}x \in K \right\} \quad (x \in V)$$

We observe the following facts:

1. Since $0$ is an internal point of $M$, the point $-x_0$ is an internal point of $M - N$ and therefore, $0$ is an internal point of $K$.

2. The set $K$ is convex by the last part of lemma ??, and it is absorbing since $0$ is an internal point of it$^3$.

$^3$See the remark in the notes from 3 November 2016, just after the “warm-up” paragraph.
3. The Minkowski functional $\mu_K$ is subadditive and positive homogeneous, the latter meaning that $\mu_K(\alpha x) = \alpha \mu_K(x)$ for all $x \in V$ and $\alpha \geq 0$. This result corresponds to theorem 1.35 in the second edition of the book “Functional Analysis” from W. Rudin.

4. The point $x_0$ is not in $K$, because if it were, there would exist $(m, n) \in M \times N$ such that $m - n = 0$, which is impossible since $M \cap N = \emptyset$.

5. If a point $x \in V$ satisfies $\mu_K(x) < 1$, then there exists a $t \in (0, 1)$ such that $t^{-1}x \in K$. This implies that $x = tk$ for some $k \in K$. But this in turn implies that $x \in K$ because $K$ is convex and contains 0. So our previous fact implies that $\mu_K(x_0) \geq 1$.

Call $\text{span}\{x_0\}$ the linear subspace of $V$ generated by $\{x_0\}$. We may define a non-zero linear functional $f : \text{span}\{x_0\} \to \mathbb{R}$ by $f(\alpha x_0) := \alpha \mu_K(x_0)$. If $\alpha > 0$, fact 3 tells us that $f(\alpha x_0) \leq \mu_K(\alpha x_0)$, and if $\alpha < 0$, then $f(\alpha x_0) = \alpha \mu_K(x_0) \leq 0 \leq \mu_K(\alpha x_0)$. So, we are exactly in the context of the Hahn-Banach theorem (real version) exposed in the notes from 8 November 2016. Call $F$ the linear extension of $f$ dominated by $\mu_K$. If $x \in K$, then $\mu_K(x) \leq 1$ by definition\(^4\). On the other hand, since $F$ agrees with $f$ on $\text{span}\{x_0\}$, fact 5 yields $F(x_0) = f(x_0) = \mu_K(x_0) \geq 1$, which finishes the proof.

3.3.28 Exercise. Prove the previous theorem in the complex case.

3.3.29 Corollary. Let $X$ be a locally convex TVS and $K_1, K_2$ two disjoint convex sets such that at least one of them has non-empty interior. Then, there exists a non-zero linear functional that separates $K_1$ and $K_2$.

Proof. If $K_1$ has an interior point $x_0 \in K_1^\circ$, then there exists a convex balanced neighborhood $U \in \mathcal{U}$ such that $x_0 + U \subseteq K_1^\circ$. Furthermore, for any $y \in X$, by continuity of scalar multiplication, there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U$. Therefore, $x_0 + (-\epsilon, \epsilon)y \subseteq x_0 + U$ and $x_0$ is an internal point of $K_1$. The result follows now from an application of the previous separation theorem ??.

3.3.30 Corollary. In a locally convex TVS $X$, the dual space $X^*$ of continuous linear functionals separates points in $X$.

Proof. By the Hausdorff property, given two distinct points $x \neq y$ in $X$, there is a convex balanced neighborhood $V \in \mathcal{U}$ such that

$$(x + V) \cap \{y\} = \emptyset,$$

so the preceding corollary applies.

In applications, it is often desirable to have strict separation, whence the following theorem.

3.3.31 Theorem. Let $V$ be a vector space and $K \subseteq V$ a convex subset, disjoint from $K$, whose points are all internal. Let $D$ be an affine subspace (i.e. $D = x + W$ for some subspace $W \subseteq V$ and point $x \in V$). Then, there exists a linear functional $f$ such that $f(D) = 0$ and $f(K) \cap \{0\} = \emptyset$.

\(^4\)Since $1^{-1}x \in K$.  

3
Proof. Up to translating both $K$ and $D$ appropriately, we may assume without loss of generality that $D$ is a linear subspace of $V$. By Masur’s separation theorem ??, there exist both a linear functional $F : V \to \mathbb{K}$ and $\beta \in \mathbb{R}$ such that,

$$
\sup \operatorname{Re} F(K) \leq \beta \leq \inf \operatorname{Re} F(D).
$$

By letting $f(x) = \operatorname{Re} F(x)$, we notice that, since $0 \in D$,

$$
\beta \leq 0 = f(0) = F(0).
$$

To be continued...