Functional Analysis, Math 7320
Lecture Notes from November 15, 2016

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Last Time

- Hahn Banach
- Separation properties

From last time:

4.2.0 Theorem. *Let V be a vector space and $K \subset V$ a convex subset whose points are all internal. Let D be an affine subspace such that $D \cap K = \emptyset$, then there is a linear functional f such that $f(D) = c$ with $c \in \mathbb{R}$ and $f(K) \subset (c, \infty)$.*

Proof. Without loss of generality assume D is a subspace, we want to show

$$f(D) = 0, \qquad f(K) \subset (0, \infty)$$

By Masur's Separation theorem, there is a linear functional F and $\beta \in \mathbb{R}$ such that

$$\sup ReF(K) \leqslant \beta \leqslant \inf ReF(D).$$

Let $f(x) = ReF(x)$, so if V is complex, then

$$F(x) = f(x) - if(ix).$$

By $0 \in D$, $\beta \leqslant f(0) = F(0)$.
Either $D = \{0\}$, and we can choose $\beta = 0$.
Next, assume there is $x \in D$ with $f(x) \neq 0$, then either $f(x) < 0$ or $f(-x) < 0$, and then

$$\inf_{\alpha \in \mathbb{R}} f(\alpha x) = -\infty,$$

contradicting $\beta \in \mathbb{R}$.
This means, we can always choose $\beta = 0$. Hence,

$$f|_D = F|_D = 0,$$

so $D \subset kerF$.
We wish to show $kerF$ and K are disjoint.

Let $x_0 \in kerF \cap K$, $y \in V$ with $f(y) > 0$. Since x_0 ($\in K$) is internal, there is $\epsilon > 0$ such that $x_0 + \epsilon y \in K$, and then by $x_0 \in kerF$,

$$f(x_0 + \epsilon y) = f(x_0) + \epsilon f(y) > 0.$$

Thus, $\sup_{x \in K} f(x) > 0$. Contradiction.
Hence, $kerF$ and K are disjoint.
So, we have that $0 \notin f(K)$, i.e. $f(K) \subset (0, \infty)$.
And D is an affine subspace which is the subset of the form

$$x + W = \{x + w : w \in W\}$$

for some $x \in V$, and W is a linear subspace of V.
For subspace W, we can get that

$$f(W) = 0, \quad f(K - x) \subset (0, \infty)$$

Let $f(x) = c$, then we have

$$f(D) = c, \quad f(K) \subset (c, \infty)$$

the proof is complete.

□

Next, we would like to strengthen the separation to a strict inequality.

4.2.1 Theorem. *Let V be a locally convex TVS and A, B disjoint non-empty convex sets. And A is compact, B is closed, then there is a continuous linear functional f such that*

$$\sup Ref(A) < \inf Ref(B).$$

Proof. Using the improved separation property of a TVS, we know there is $U \in \mathbb{U}$ open, convex and balanced such that

$$(A + U) \cap (B + U) = \infty,$$

which $A + U$ is open and convex.
By the corollary to Masur on locally convex TVS, there is a continuous non-zero linear functional f such that

$$\sup Ref(A + U) \leqslant \inf Ref(B + U).$$

Pick $x \in U$ such that $f(x) = \epsilon > 0$, then

$$\begin{aligned} \sup Ref(A + x) &\leqslant \sup Ref(A + U) \\ &\leqslant \inf Ref(B + U) \\ &\leqslant \inf Ref(B - x) \end{aligned}$$

By the linearity of f,

$$\sup Ref(A) + \epsilon \leqslant \inf Ref(B) - \epsilon$$

hence,

$$\sup Ref(A) < \inf Ref(B).$$

□

4.3 The Weak Topology of X

4.3.2 Question. Assume we forgot the topology of X and only know X^*. What do we know about the topology of X?
We could use X^* to define initial topology on X.
Does this change the set of linear continuous functionals?

4.3.3 Remark. Let X be a real or complex vector space, and F a collection of linear functionals $X \to Y$.
The sets of the form

$$\{y \in X : |f(y) - f(x)| < \epsilon\}$$

where $x \in X$, $\epsilon > 0$ and $f \in F$ vary, is a subbase for a topology on X, namely the topology where a subset of X is open if and only if it is the union of sets which are the intersection of a finite collection of such sets.
This is called the F-topology of X.

4.3.4 Lemma. *The F-topology is Hausdorff if and only if F separates the points of X.*

Proof. Let x_0, $y_0 \in X$, $x_0 \neq y_0$. If the F-topology is Hausdorff there are open set U, V such that $x_0 \in U$, $y_0 \in V$ and $U \cap V = \emptyset$.
We may assume that U and V are intersections of finite collections of sets of the form

$$\{y \in X : |f(y) - f(x)| < \epsilon\}$$

It follows that there is a set of that form which contains x_0 but not y_0. I.e.

$$x_0 \in \{y \in X : |f(y) - f(x)| < \epsilon\}$$

while $|f(y_0) - f(x)| \geq \epsilon$ for some $x \in X$, $f \in F$ and some $\epsilon > 0$.
Then $f(x_0) \neq f(y_0)$, and we conclude that F separates the points of X.
Conversely, assume that F separates the points of X.
Let x_0, $y_0 \in X$, $x_0 \neq y_0$.
There is then a functional $f \in F$ such that $f(x_0) \neq f(y_0)$.
Set $\epsilon = \frac{1}{2}|f(x_0) - f(y_0)| > 0$, and note that

$$x_0 \in \{y \in X : |f(y) - f(x_0)| < \epsilon\}$$

$$y_0 \in \{y \in X : |f(y) - f(y_0)| < \epsilon\}$$

Since

$$\{y \in X : |f(y) - f(y_0)| < \epsilon\} \cap \{y \in X : |f(y) - f(x_0)| < \epsilon\} = \emptyset$$

the proof is complete. □

4.3.5 Lemma. *Let $f_1, f_2, \dots, f_n$ be linear functionals on a vector space V. Let*

$$N = ker f_1 \cap ker f_2 \cap \dots \cap ker f_n,$$

then the following three properties are equivalent for a linear functional f:
(1) $\quad f \in Span\{f_1, f_2, \dots, f_n\}$
(2) There is $C > 0$ such that for all $x \in V$,

$$|f(x)| \leqslant C \max_k |f_k(x)|$$

(3) $\quad N \subset ker f$.

Proof. Assume (1), i.e. there is $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$f(x) = \sum_{k=1}^{n} \alpha_k f_k(x)$$

then

$$|f(x)| \leqslant \sum_{k=1}^{n} |\alpha_k||f_k(x)| \leqslant n(\max_{1\leqslant k\leqslant n} |\alpha_k|) \max_{1\leqslant k\leqslant n} |f_k(x)|$$

where $C = n(\max_{1\leqslant k\leqslant n} |\alpha_k|)$.
Assume (2) holds, so

$$|f(x)| \leqslant C \max_k |f_k(x)|$$

then f vanishes on N.
Assume (3) holds. Let $\mathbb{F} = \mathbb{R} or \mathbb{C}$ be the ground field for V.
Let $T: \quad V \to \mathbb{F}^n$,

$$T(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

If $x, y \in V$ give $T(x) = T(y)$, then $x - y \in N$ and $f(x-y) = 0$ by assumption, so $f(x) = f(y)$.
Let $\Lambda: \quad T(V) \subset \mathbb{F}^n \to \mathbb{F}$,

$$\Lambda((f_1(x), f_2(x), \dots, f_n(x))) = f(x)$$

then Λ is linear and it extends linearly to all of $\mathbb{F}^n$.
Hence, there are $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ with

$$\Lambda(u_1, u_2, \dots, u_n) = \sum_{j=1}^{n} \alpha_j u_j$$

Consequently, $f(x) = \Lambda((f_1(x), f_2(x), \dots, f_n(x))) = \sum_{j=1}^{n} \alpha_j f_j(x)$.

□

4.3.6 Theorem. *Let V be a vector space, V' a separating vector space of linear functionals on V. Denote by τ' the initial topology induced by V' on V, then (V, τ') is a locally convex TVS and the space of all linear continuous functionals is V'.*

Proof. Since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is Hausdorff, and V' separates points, (V, τ') is Hausdorff by the previous $4.3.4$ $Lemma$. The topology τ' is translation invariant because open sets in (V, τ') are

generated by $\{f^{-1}(A) : f \in V', A \text{ open in } \mathbb{F}\}$ and f is linear.
Hence we have a local subbase

$$V(f,r) = \{x \in V : |f(x)| < r\}$$

whose sets are convex and balanced. Moreover, since V' separates points,

$$\bigcap_{r>0, f \in V'} V(f,r) = \{0\}$$

so the singleton set is closed. (Next part of proof see the next lecture notes)

□