Last Time

- Hahn Banach
- Separation properties

From last time:

4.2.0 Theorem. Let $V$ be a vector space and $K \subset V$ a convex subset whose points are all internal. Let $D$ be an affine subspace such that $D \cap K = \emptyset$, then there is a linear functional $f$ such that $f(D) = c$ with $c \in \mathbb{R}$ and $f(K) \subset (c, \infty)$.

Proof. Without loss of generality assume $D$ is a subspace, we want to show

$$f(D) = 0, \quad f(K) \subset (0, \infty)$$

By Masur’s Separation theorem, there is a linear functional $F$ and $\beta \in \mathbb{R}$ such that

$$\sup_{x \in K} ReF(x) \leq \beta \leq \inf_{x \in D} ReF(x).$$

Let $f(x) = ReF(x)$, so if $V$ is complex, then

$$F(x) = f(x) - if(ix).$$

By $0 \in D$, $\beta \leq f(0) = F(0)$.
Either $D = \{0\}$, and we can choose $\beta = 0$.
Next, assume there is $x \in D$ with $f(x) \neq 0$, then either $f(x) < 0$ or $f(-x) < 0$, and then

$$\inf_{\alpha \in \mathbb{R}} f(\alpha x) = -\infty,$$

contradicting $\beta \in \mathbb{R}$.
This means, we can always choose $\beta = 0$. Hence,

$$f|_D = F|_D = 0,$$

so $D \subset kerF$.
We wish to show $kerF$ and $K$ are disjoint.
Let \( x_0 \in ker F \cap K \), \( y \in V \) with \( f(y) > 0 \). Since \( x_0 (\in K) \) is internal, there is \( \epsilon > 0 \) such that \( x_0 + \epsilon y \in K \), and then by \( x_0 \in ker F \),

\[
f(x_0 + \epsilon y) = f(x_0) + \epsilon f(y) > 0.
\]

Thus, \( \sup_{x \in K} f(x) > 0 \). Contradiction.

Hence, \( ker F \) and \( K \) are disjoint.

So, we have that \( 0 \notin f(K) \), i.e. \( f(K) \subset (0, \infty) \).

And \( D \) is an affine subspace which is the subset of the form

\[
x + W = \{ x + w : w \in W \}
\]

for some \( x \in V \), and \( W \) is a linear subspace of \( V \).

For subspace \( W \), we can get that

\[
f(W) = 0, \quad f(K - x) \subset (0, \infty)
\]

Let \( f(x) = c \), then we have

\[
f(D) = c, \quad f(K) \subset (c, \infty)
\]

the proof is complete.

Next, we would like to strengthen the separation to a strict inequality.

4.2.1 Theorem. Let \( V \) be a locally convex TVS and \( A, B \) disjoint non-empty convex sets. And \( A \) is compact, \( B \) is closed, then there is a continuous linear functional \( f \) such that

\[
\sup Ref(A) < \inf Ref(B).
\]

Proof. Using the improved separation property of a TVS, we know there is \( U \in \mathcal{U} \) open, convex and balanced such that

\[
(A + U) \cap (B + U) = \infty,
\]

which \( A + U \) is open and convex.

By the corollary to Masur on locally convex TVS, there is a continuous non-zero linear functional \( f \) such that

\[
\sup Ref(A + U) \leq \inf Ref(B + U).
\]

Pick \( x \in U \) such that \( f(x) = \epsilon > 0 \), then

\[
\sup Ref(A + x) \leq \sup Ref(A + U) \leq \inf Ref(B + U) \leq \inf Ref(B - x)
\]

By the linearity of \( f \),

\[
\sup Ref(A) + \epsilon \leq \inf Ref(B) - \epsilon
\]

hence,

\[
\sup Ref(A) < \inf Ref(B).
\]
4.3 The Weak Topology of $X$

4.3.2 Question. Assume we forgot the topology of $X$ and only know $X^*$. What do we know about the topology of $X$?
We could use $X^*$ to define initial topology on $X$.
Does this change the set of linear continuous functionals?

4.3.3 Remark. Let $X$ be a real or complex vector space, and $F$ a collection of linear functionals $X \to Y$.
The sets of the form
\[ \{ y \in X : |f(y) - f(x)| < \epsilon \} \]
where $x \in X$, $\epsilon > 0$ and $f \in F$ vary, is a subbase for a topology on $X$, namely the topology where a subset of $X$ is open if and only if it is the union of sets which are the intersection of a finite collection of such sets.
This is called the $F$-topology of $X$.

4.3.4 Lemma. The $F$-topology is Hausdorff if and only if $F$ separates the points of $X$.

Proof. Let $x_0, y_0 \in X$, $x_0 \neq y_0$. If the $F$-topology is Hausdorff there are open set $U, V$ such that $x_0 \in U$, $y_0 \in V$ and $U \cap V = \emptyset$.
We may assume that $U$ and $V$ are intersections of finite collections of sets of the form
\[ \{ y \in X : |f(y) - f(x)| < \epsilon \} \]
It follows that there is a set of that form which contains $x_0$ but not $y_0$. I.e.
\[ x_0 \in \{ y \in X : |f(y) - f(x)| < \epsilon \} \]
while $|f(y_0) - f(x)| \geq \epsilon$ for some $x \in X$, $f \in F$ and some $\epsilon > 0$.
Then $f(x_0) \neq f(y_0)$, and we conclude that $F$ separates the points of $X$.
Conversely, assume that $F$ separates the points of $X$.
Let $x_0, y_0 \in X$, $x_0 \neq y_0$.
There is then a functional $f \in F$ such that $f(x_0) \neq f(y_0)$.
Set $\epsilon = \frac{1}{2}|f(x_0) - f(y_0)| > 0$, and note that
\[ x_0 \in \{ y \in X : |f(y) - f(x_0)| < \epsilon \} \]
\[ y_0 \in \{ y \in X : |f(y) - f(y_0)| < \epsilon \} \]
Since
\[ \{ y \in X : |f(y) - f(y_0)| < \epsilon \} \cap \{ y \in X : |f(y) - f(x_0)| < \epsilon \} = \emptyset \]
the proof is complete.

4.3.5 Lemma. Let $f_1, f_2, \ldots, f_n$ be linear functionals on a vector space $V$. Let
\[ N = \ker f_1 \cap \ker f_2 \cap \cdots \cap \ker f_n, \]
then the following three properties are equivalent for a linear functional $f$:

1. $f \in \text{Span}\{f_1, f_2, \ldots, f_n\}$
2. There is $C > 0$ such that for all $x \in V$,
   $$|f(x)| \leq C \max_k |f_k(x)|$$
3. $N \subset \ker f$.

Proof. Assume (1), i.e. there is $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that
   $$f(x) = \sum_{k=1}^{n} \alpha_k f_k(x)$$
then
   $$|f(x)| \leq \sum_{k=1}^{n} |\alpha_k||f_k(x)| \leq n(\max_{1 \leq k \leq n} |\alpha_k|) \max_{1 \leq k \leq n} |f_k(x)|$$
where $C = n(\max_{1 \leq k \leq n} |\alpha_k|)$.
Assume (2) holds, so
   $$|f(x)| \leq C \max_k |f_k(x)|$$
then $f$ vanishes on $N$.
Assume (3) holds. Let $\mathbb{F} = \mathbb{R} or \mathbb{C}$ be the ground field for $V$.
Let $T : V \to \mathbb{F}^n$,
   $$T(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$
If $x, y \in V$ give $T(x) = T(y)$, then $x - y \in N$ and $f(x - y) = 0$ by assumption, so $f(x) = f(y)$.
Let $\Lambda : T(V) \subset \mathbb{F}^n \to \mathbb{F}$,
   $$\Lambda((f_1(x), f_2(x), \ldots, f_n(x))) = f(x)$$
then $\Lambda$ is linear and it extends linearly to all of $\mathbb{F}^n$.
Hence, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ with
   $$\Lambda(u_1, u_2, \ldots, u_n) = \sum_{j=1}^{n} \alpha_j u_j$$
Consequently,
   $$f(x) = \Lambda((f_1(x), f_2(x), \ldots, f_n(x))) = \sum_{j=1}^{n} \alpha_j f_j(x).$$

4.3.6 Theorem. Let $V$ be a vector space, $V'$ a separating vector space of linear functionals on $V$. Denote by $\tau'$ the initial topology induced by $V'$ on $V$, then $(V, \tau')$ is a locally convex TVS and the space of all linear continuous functionals is $V'$.

Proof. Since $\mathbb{F} = \mathbb{R} or \mathbb{C}$ is Hausdorff, and $V'$ separates points, $(V, \tau')$ is Hausdorff by the previous 4.3.4 Lemma. The topology $\tau'$ is translation invariant because open sets in $(V, \tau')$ are
generated by \( \{ f^{-1}(A) : f \in V', A \text{ open in } \mathbb{R} \} \) and \( f \) is linear.

Hence we have a local subbase

\[
V(f, r) = \{ x \in V : |f(x)| < r \}
\]

whose sets are convex and balanced. Moreover, since \( V' \) separates points,

\[
\bigcap_{r > 0, f \in V'} V(f, r) = \{ 0 \}
\]

so the singleton set is closed. (Next part of proof see the next lecture notes)