# Functional Analysis, Math 7320
# Lecture Notes from November 15, 2016

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**3.2.13 Theorem.** *let $V$ is vector space, $K \subset V$ convex, all points in $K$ are interior. $D$ is affine subspace, $D \bigcap K = \emptyset$, then there is linear function $f, c \in \mathbb{R}$ with $f(D) = c$, $f(K) \subset (c, \infty)$.*

*Proof.* without lost of generality, let $D$ is a subspace,want to show $f(D) = 0$, $f(K) \subset (0, \infty)$.

by Mazur's separation theorem, we had $F, \beta \in \mathbb{R}$, with

$$\sup \Re F(K) \leq \beta \leq \inf \Re F(D)$$

. let $f(x) = \Re F(x)$, if $V$ is complex, $F(x) = f(x) - if(ix)$. By $0 \in D$, $\beta \leq f(0) = F(0)$, Either $D = \{0\}$, and we can choose $\beta = 0$.

Next, assume there is $x \in D$ with $f(x) \neq 0$. then

$$\text{either } f(x) < 0 \text{ or } f(-x) < 0$$

and then

$$\inf_{\alpha \in \mathbb{R}} f(\alpha x) = -\infty$$

which contradicting $\beta \in \mathbb{R}$

this means that we can always choose $\beta = 0$. Hence,

$$f|_D = F|_D = 0, \text{ so } D \subset \ker F$$

we wish to show $\ker F$ and $K$ is disjoined:

Let $x_0 \in K \bigcap \ker F$, $y \in V$ with $f(y) > 0$. since $x_0$ is interior in $K$, there is $\epsilon > 0$ s.t. $x_0 + \epsilon y \in K$, and then by $x_0 \in \ker F$

$$f(x_0 + \epsilon y) = f(x_0) + \epsilon f(y) > 0$$

. Thus, $\sup_{x \in K} f(x) > 0$, contradiction! so $\ker F$ and $K$ are disjoined.

if $D$ is not subspace, that means $D = a + D'$ where $a$ is a translation and $D'$ is subspace.

then apply the argument to $D'$, we have

$$f(D') = 0, f(K - a) \subset (0, \infty)$$

then let $c = f(a)$, we have

$$f(D) = c, f(K) \subset (c, \infty)$$

□

next, we would like to strengthen the separation to a strict inequality:

**3.2.14 Theorem.** *let $V$ be a locally convex topological vector space, and $A, B$ disjoint non-empty convex sets, $A$ compact, $B$ closed, then there is a continuous linear function $f$, s.t.*

$$\sup \Re f(A) < \inf \Re f(B).$$

*Proof.* using the improved separation property of topological vector space, we know there is $U \in \mathcal{U}$ open convex and balanced, s.t.

$$(A+U) \bigcap (B+U) = \emptyset$$

note that $A+U$ is still open and convex, by the corollary to Mazur on local convex topological vector space, there is a continuous non-zero linear function $f$ s.t

$$\sup \Re f(A+U) \leq \inf \Re f(B+U)$$

Pick $x \in U$ s.t $f(x) = \epsilon > 0$ (if $f(x) < 0$, then take $-x$). Then

$$\sup \Re f(A+x) \leq \sup \Re f(A+U) \leq \inf \Re f(B+U) \leq \inf \Re f(B-x)$$

by the linearity of $f$,

$$\sup \Re f(A) + \epsilon \leq \inf \Re f(B) - \epsilon$$

here

$$\sup \Re f(A) < \inf \Re f(B)$$

□

## 3.3 the weak topology of $X$

Question: Assume we forgot the topology of $X$, and only know $X^*$, What do we know about the topology of $X$? we could use $X^*$ to define initial topology on $X$, does this change the set of linear continuous functions?

**3.3.15 Lemma.** *Let $f_1, f_2, \ldots, f_n$ be linear functions on a vector space $V$. let $N = \ker f_1 \bigcap \ker f_2 \bigcap \ldots \ker f_n$, then the following are equivalent for a linear function $f$:*

*(1) $f \in$ span$\{f_1, f_2, \ldots, f_n\}$*

*(2) there is $C > 0$ which relies on $f$, s.t for all $x \in V$, $|f(x)| \leq C \max_k |f_k(x)|$*

*(3) $N \subset \ker f$*

*Proof.* Assume (1), i.e there is $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$f(x) = \sum_{k=1}^{n} \alpha_k f_k(x),$$

then

$$|f(x)| \leq \sum_{k=1}^{n} |\alpha_k||f_k(x)| \leq n(\max_{1\leq k\leq n} |\alpha_k|) \max_{1\leq k\leq n} |f_k(x)|$$

and we denote $C = n \max_{1\leq k\leq n} |\alpha_k|$.

Assume (2) holds, so $|f(x)| \leq C \max_k |f(x)|$, then $f$ vanishes on $N$.

Assume(3) holds, Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be the number filed for $V$, let $T : V \to \mathbb{F}^n$,

$$T(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

let $x, y \in V$, if $T(x) = T(y)$, then $x - y \in N$, so $f(x - y) = 0$ by assumption, so $f(x) = f(y)$.

so the following map is well defined: $\wedge : T(V) \subset \mathbb{F}^n \to \mathbb{F}$:

$$\wedge(f_1(x), f_2(x), \ldots, f_n(x)) = f(x)$$

the $\wedge$ is linear and it extends linearly to all of $\mathbb{F}^n$. Hence, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ with

$$\wedge(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \alpha_j u_j.$$

consequently,

$$f(x) = \wedge(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} \alpha_i f_i(x)$$

□

**3.3.16 Theorem.** *let $V$ be a vector space, $V^{'}$ a separating vector space of linear functions on $V$. denote by $\tau'$ the initial topology induced by $V'$ on $V$, then $(V, \tau')$ is a locally convex topological vector space and the space of all linear continuous function is $V'$.*

*Proof.* since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is Hausdorff, and $V'$ separate points. $(V, \tau')$ is Hausdorff, the topology $\tau'$ is translation invariant because open sets in $(V, \tau')$ are generated by

$$\{f^{-1}(A) : f \in V', A \text{ open in } \mathbb{F}\}, \text{and } f \text{ is linear}$$

Hence we have a local subbase:

$$V(f, r) = \{x \in V, |f(x)| < r\}$$

where sets are convex and balanced.

moreover, since $V'$ separeates points, so

$$\bigcap_{r>0, f\in V'} V(f, r) = \{0\}.$$

because

if not, i.e. there is non zero $x$,

$$x \in \bigcap_{r>0, f\in V'} V(f, r)$$

it means for any $f \in V'$ $r > 0$, we have$|f(x)| < r$
then by separation, there exists $f \in V'$ and $r \in \mathbb{R}$, s.t

$$f(0) < r < f(x)$$

a contradiction! so

$$\bigcap_{r>0, f \in V'} V(f, r) = \{0\}.$$

so the singleton set is closed. $\square$