4.1.6 Theorem. Suppose $V$ is a vector space and $V'$ is a separating vector space of linear functionals on $V$. Denote that $\tau'$ is the initial topology induced by $V'$ on $V$. Then $(V, \tau')$ is locally convex topological space, and the space of all continuous linear functional is $V'$.

Proof. Last time, we obtain local convex sub-base, and $0$ is closed. Next for $f \in V'$ and if $r > 0$ we have

$$\frac{1}{2}V(f, r) + \frac{1}{2}V(f, r) = \{ \frac{1}{2}x + \frac{1}{2}y : |f(x)| < r, |f(y)| < r \} \subseteq V(f, r).$$

By the same argument with finite intersections, every element in local base formed from the sub-base satisfies the same inclusion under scaling and addition. This shows that addition is continuous at $0$, and by translation invariant everywhere. By the definition of initial topology, each $f \in V'$ is continuous with respect to $\tau$. Conversely, given a $\tau$-continuous linear functional $f$ on $V$. Then $f^{-1}(B_1(0))$ is open, or there is $U$ in local base such that $|f(U)| \subseteq [0, 1)$, and $\ker f \subseteq U$. In particular, there are continuous linear functionals \{ $f_1, ..., f_n$ \} $r_1 ... r_n > 0$ with

$$U = \{ x \in V : |f_1(x)| < r_1, ..., |f_n(x)| < r_n \}$$

and

$$\sup_{x \in U} |f(x)| \leq 1$$

By linearity, $\varepsilon U = \{ x \in X : |f_1(x)| < \varepsilon r_1, ..., |f_n(x)| < \varepsilon r_n \}$ gives

$$\sup_{x \in U} |f(x)| < \varepsilon$$

Taking intersections,

$$\ker f \subseteq \bigcap_{\varepsilon > 0} (\varepsilon U)$$

$$= \ker(f_1) \cap ... \cap \ker(f_n)$$

Hence, by lemma, $f \in \text{span} \{ f_1, ..., f_m \}$ so $f \in V'$.

4.1.7 Example. In $\mathbb{R}^n$, let $V'$ be the set of all linear functionals on $\mathbb{R}^n$. If $x \neq y$, there is $i \in \{ 1, ..., n \}$ such that $x_i \neq y_i$. Define $f_i(x_1, ..., x_i, ..., x_n) = x_i$. Then, $f_i \in V'$ and $x_i = f_i(x) \neq f_i(y) = y_i$. Thus, $V'$ separates points in $\mathbb{R}^n$. Moreover, the initial topology induced by $V'$ is the standard topology on $\mathbb{R}^n$. 

$\blacksquare$
The Weak Topology of a Topological Vector Space

Recall that the dual space $X^*$ of $X$ is the set of all continuous linear functionals on $X$, i.e.,

$$X^* = \{ f : X \to K : f \text{ is linear continuous} \}$$

where $K$ is either $\mathbb{R}$ or $\mathbb{C}$.

4.1.8 Remark. From the previous theorem, if $X^*$ separates points on $X$, the initial topology generated by $X^*$ is a topological vector space. However, the dual space $X^*$ might not separate points of $X$. For example, the space $L^p[0, 1]$ where $0 < p < 1$ has only $0$ as a continuous linear functional. The topology induced by continuous linear functional is not a topological vector space. We are interested in topological vector spaces. To make sure that the initial topology induced by $X^*$ is a topological vector space, we need to include the condition of separating points of $X^*$.

Proof. To prove the example mentioned above for $0 < p < 1$, $L^p[0, 1]^* = \{ 0 \}$, we use contradiction. Assume there exists $\varphi \in L^p[0, 1]^*$ with $\varphi \neq 0$. Then $\varphi$ has image $\mathbb{R}$ (a nonzero linear map to a one-dimensional space is surjective), so there is some $f \in L^p[0, 1]$ such that $|\varphi(f)| \geq 1$. Using this choice of $f$, map $[0, 1]$ to $\mathbb{R}$ by

$$s \mapsto \int_0^s |f(x)|^p \, dx$$

This is continuous, so there is some $s$ between $0$ and $1$ such that

$$\int_0^s |f(x)|^p \, dx = \frac{1}{2} \int_0^1 |f(x)|^p \, dx > 0.$$ 

Now let $g_1 = f|_{[0,s]}$ and $g_2 = f|_{(s,1)}$, so $f = g_1 + g_2$ and $|f|^p = |g_1|^p + |g_2|^p$. So

$$\int_0^1 |g_1(x)|^p \, dx = \int_0^s |f(x)|^p \, dx = \frac{1}{2} \int_0^1 |f(x)|^p \, dx,$$

hence $\int_0^1 |g_2(x)|^p \, dx = \frac{1}{2} \int_0^1 |f(x)|^p \, dx$. Since $|\varphi(f)| \geq 1$, $|\varphi(g_i)| \geq \frac{1}{2}$ for some $i$. Let $f_1 = g_i$, so $|\varphi(f_1)| \geq 1$ and $\int_0^1 |f_1(x)|^p \, dx = \int_0^1 |g_i(x)|^p \, dx = 2^{p-1} \int_0^1 |f(x)|^p \, dx$. Note that $2^{p-1} < 1$ by iteration, we get a sequence $f_n$ in $L^p[0, 1]$ such that $|\varphi(f_n)| \geq 1$ and

$$d(f_n, 0) = \int_0^1 |f_n(x)|^p \, dx = (2^{p-1})^n \int_0^1 |f(x)|^p \, dx \to 0,$$

a contradiction of continuity of $\varphi$.

\[ \square \]

4.1.9 Definition. Let $(X, \tau)$ be a topological vector space whose dual $X^*$ separates points on $X$. Then the initial topology induced by $X^*$ on $X$ is called the weak topology and denoted by $\tau_w$.

4.1.10 Corollary. $X_w$ is a locally convex topological vector space and $X_w^* = X^*$.
4.1.11 Corollary. $(X_w)_w = X_w$

4.1.12 Remark. As we know, the weak topology $\tau_w$ on $X$ induced by $X^*$ is the coarsest topology which any $f \in X^*$ is continuous. By the definition of $X^*$, any $f \in X^*$ is continuous with respect to the original topology $\tau$. Therefore, $\tau_w \subseteq \tau$. Since a finite dimensional real (or complex) vector space is homeomorphism to $\mathbb{R}^n$ (or $\mathbb{C}^n$), the weak topology and the original topology in finite dimensional vector space are the same. The question that we should ask is that "in general, are the weak topology and the original topology in a topological vector space the same?", and "if they are not the same, do they share same properties?".

In general, if $f : X \to Y$ is continuous and $x_n \to x$, we have $f(x_n) \to f(x)$. Moreover, if $X$ is first countable, we have that $f$ is continuous if and only if $f(x_n) \to f(x)$ for every $x_n \to x$. Next theorem characterizes the weak convergence of a sequence in $X$ by continuous linear functionals.

Warm up

4.1.13 Proposition. A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological vector space weakly converges to zero, $x_n \overset{w}{\to} 0$, if and only if for each $f \in X^*$, $f(x_n) \to 0$.

Proof. For each $\tau_w$-neighborhood of 0, there is $N \in \mathbb{N}$ such that for all $n \geq N, x_n \in V$. Given $f \in X^*$ and $\epsilon > 0$, then by weak convergence of $(x_n)_{n \in \mathbb{N}}$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V(f, \epsilon)$, so $|f(x_n)| < \epsilon$ hence $f(x_n) \to 0$. Conversely, if $f(x_n) \to 0$ for each $f \in V^*$, then we have that for each $\{f_1, ..., f_m\} \subset X^*$ there is $N \in \mathbb{N}$ for which if $n \geq N$, $x_n \in V(f_1, r_1) \cap V(f_2, r_2) \cap ... \cap V(f_m, r_m)$ for every $U \in \mathcal{U}$, we can find $\{f_1, ..., f_m\}, r_1, ..., r_m$ such that the intersection is in $U$. Hence, we get $x_n \overset{w}{\to} 0$.

4.1.14 Corollary. Strong convergence with respect to $\tau$ (i.e. convergence of a sequence with respect to $\tau$) implies convergence with respect to $\tau_w$.

Proof. If $x_n \to 0$, then for $f \in X^*$, $f(x_n) \to 0$, so the preceding proposition applies.

4.1.15 Remark. The converse of the previous corollary is not true. For instance, in $L^p = L^p(-\pi, \pi)$ with respect to Lebesgue measure and $p \geq 1$. Define $f_n(t) = e^{int}$. Then, $f_n \to 0$ weakly in $L^p$ but not strongly.

Next, we study boundedness in the weak topology

4.1.16 Proposition. Let $(X, \tau)$ be a topological vector space. A set $E$ is $\tau_w$ bounded if and only if for each linear functional $f \in X^*$, then $f$ is bounded on $E$.

Proof. $E$ is weakly bounded if and only if for each $\tau_w$-neighborhood $V$ of 0, there exist $s \geq 0$ such that for all $t > 0$, $E \subset tv$. This is equivalent to the fact that for every set of the form $\{x \in X : |f_j(x)| < r_j, 1 \leq j \leq m\}$ and for all $t > s$,

$$E \subseteq \{tx \in X : |f(x)| < r_j, 1 \leq j \leq m\}$$
\[ y \in X : |f(y)| < tr_j, 1 \leq j \leq m \]

imply all \( f_j' \)'s are bounded on \( E \).

4.1.17 Proposition. If \((X, \tau)\) is an infinite-dimensional topological vector space, then every \( \tau_w \)-neighborhood of \( 0 \) contains an infinite dimensional subspace. In particular, \((X, \tau_w)\) is not locally bounded.(no chance for countable local base)

Proof. Given \( U \in \mathcal{U} \), then there exists \( V \subset U \) of the form

\[ V = \{ x \in X : |f_j(x)| < r_j, 1 \leq j \leq m \} \]

and so is \( N = \ker f_1 \cap \ldots \cap \ker f_n \). Thus, \( x \mapsto (f_1(x), f_2(x), \ldots, f_m(x)) \) has the null space \( N \). By dimension counting, \( \dim X \leq m + \dim N \), this implies \( \dim N = \infty \).

Closedness in Weak Topology

4.1.18 Remark. If \( E \) is \( \tau_w \)-closed, then by \( E \subset \bar{E} \subset \bar{E}^w = E \). Then \( E \) is closed in \( \tau \).(this is because \( \tau_w \) is coarser than the original topology).

Q: When is the reverse \( E^w \subset E \) true?