2.1 Weak Topology vs. Original Topology (cont.)

Last time we defined the weak topology $\tau_w$ on a TVS $X$ whose dual $X^*$ separates points as the coarsest topology such that all elements of $X^*$ are continuous. We then examined the relationships between the topological properties of $(X, \tau_w)$ and $(X, \tau)$. Regarding closedness, we remarked that because $\tau_w \subset \tau$, the $\tau$-closure of any set $E$ is contained in the $\tau_w$-closure of $E$, i.e. $E \subset E^w$. Now we show that in a locally convex TVS, the reverse inclusion is also true.

2.1.1 Remark. The weak topology of a TVS $X$ was only defined in the case that $X^*$ separates points. However, recall that if $X$ is a locally convex TVS, then $X^*$ separates points (this was a corollary to Masur’s theorem). Hence it always makes sense to talk about the weak topology on a locally convex TVS.

2.1.2 Theorem. Let $E$ be a convex subset of a locally convex TVS $X$. Then $E = E^w$.

Proof. Recall a corollary to Masur’s Theorem: if $A$ and $B$ are disjoint nonempty convex sets in a locally convex TVS with $A$ compact and $B$ closed, then there is a continuous linear functional $f$ such that $\sup \text{Re } f(A) < \beta < \inf \text{Re } f(B)$ for some $\beta \in \mathbb{R}$.

Note that if $E = X$, then since $E \subset E^w$ we have $E^w = X$, hence $E = E^w$. So we proceed with the case when $E \neq X$.

Let $x_0 \not\in \overline{E}$. Since $\{x_0\}$ is compact and $\overline{E}$ is closed, there exists a continuous linear functional $f$ such that $\text{Re } f(x_0) < \beta < \inf \text{Re } f(E)$ for some $\beta \in \mathbb{R}$. Then the set $\{x \in X : \text{Re } f(x) < \beta\}$ is a weakly open neighborhood of $x_0$ which does not intersect $E$. So $x_0 \not\in \overline{E}^w$. Thus $\overline{E}^w \subset E$ by taking complements. Since the reverse inclusion holds in general, we have equality. \(\square\)

This theorem yields a simple corollary that helps us further understand closedness in the weak topology of a locally convex TVS.

2.1.3 Corollary. For a convex subset $E$ of a locally convex TVS:

1. $E$ is $\tau$-closed iff $E$ is $\tau_w$-closed.

2. $E$ is $\tau$-dense iff $E$ is $\tau_w$-dense.
Proof. For (1), note that $E$ is $\tau$-closed iff $E = \overline{E}$. But this happens iff $E = \overline{E}^w$ since $E = \overline{E}^w$ by the theorem. Finally, we have $E = \overline{E}^w$ iff $E$ is $\tau_w$-closed.

For (2), we have $E$ is $\tau$-dense iff $X = \overline{E}$. Again, using the theorem this happens iff $X = \overline{E}^w$ which is equivalent to saying $E$ is $\tau_w$-dense. \hfill $\square$

In a metrizable space, we can characterize closed sets entirely in terms of sequences (since in this setting a point $x$ is in $\overline{E}$ iff there exists a sequence of points in $E$ which converges to $x$). Viewing the above theorem with this lens yields the following consequence for sequences.

**2.1.4 Corollary.** If $E$ is a convex set in a metrizable locally convex TVS $X$, and $(x_n)_{n \in \mathbb{N}}$ converges weakly to $x \in X$, then there is a sequence $(y_n)_{n \in \mathbb{N}} \subset E$ such that $y_n \to x$ in the original topology of $X$.

**Proof.** Since $x_n \to x$ w.r.t. the weak topology, $x \in \overline{E}^w$. But by the above theorem, $\overline{E} = \overline{E}^w$, hence $x \in \overline{E}$. Thus since $X$ is metrizable, there is a sequence $(y_n)_{n \in \mathbb{N}} \subset E$ which converges to $x$ w.r.t. the original topology. \hfill $\square$

## 2.2 The Weak-* Topology

**2.2.5 Question.** So far we have treated $X^*$ as a vector space with no additional structure. If $X$ is a Banach space, we could equip $X^*$ with the operator norm to make it a Banach space as well. What should we do if $X$ is a TVS with less structure?

**2.2.6 Answer.** We may use the linear functionals on $X^*$, i.e. the elements in $X^{**}$ to define an initial topology on $X^*$. Recall from linear algebra that there is a natural identification of $X^{**}$ with $X$ given by the map $i : X \to X^{**}$ defined by $i(x) = F_x$, where $F_x(f) = f(x)$ for all $f \in X^*$ ($F_x$ evaluates the functionals in $X^*$ at $x$).

Note that $F_x(\alpha f + g) = (\alpha f + g)(x) = \alpha f(x) + g(x) = \alpha F_x(f) + F_x(g)$, so these maps are in fact linear functionals on $X^*$. We also know that $\{F_x\}_{x \in X}$ separates points in $X^*$, because if $f, g \in X^*$ and $f(x) = g(x)$ for all $x \in X$, then $f = g$. Hence $\{F_x\}_{x \in X}$ induces an initial topology on $X^*$ called the weak-* topology of $X^*$.

Note that for this definition we don’t care if $X^*$ separate points of $X$, since $\{F_x\}_{x \in X}$ always separates points of $X^*$.

**2.2.7 Remark.** In particular, the weak-* topology turns $X^*$ into a locally convex TVS, and every weak-* continuous linear functional on $X^*$ must actually equal $F_x$ for some $x \in X$ (we proved this last time before defining the weak topology).

Since the weak-* topology on $X^*$ is defined as an initial topology, we can characterize its open sets in terms of a local subbase of “balls”.

**2.2.8 Remark.** The weak-* topology on $X^*$ is generated by the local subbase $\{V(x, r)\}_{x \in X, r > 0}$ where:

$$V(x, r) = \{f \in X^* : |F_x(f)| < r\} = \{f \in X^* : |f(x)| < r\}.$$
This characterization of weak-* open sets shows that weak-* convergence of a sequence of linear functionals is equivalent to pointwise convergence (in other words, the weak-* topology can be thought of as the topology of pointwise convergence on $X^*$).

2.2.9 Proposition. Let $X$ be a TVS, and $X^*$ be equipped with the weak-* topology. Given a sequence $(f_n)_{n \in \mathbb{N}}$ in $X^*$ and $f \in X^*$, then $f_n \to f$ iff for each $x \in X$, $\lim_{n \to \infty} f_n(x) = f(x)$.

Proof. Suppose $f_n \to f$, i.e. $f_n - f \to 0$. Let $x \in X$ and fix $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $f_n - f \in V(x, \epsilon)$ for all $n \geq N$. This means that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we have $f_n(x) \to f(x)$.

Conversely, suppose $f_n(x) \to f(x)$ for all $x \in X$. Let $U \subset X^*$ be a neighborhood of 0, then there exists a basis element $\bigcap_{i=1}^{m} V(x_i, r_i) \subset U$ which contains 0. For each $i = 1, \ldots, m$, there exists an $n_i \in \mathbb{N}$ such that $|f_n(x_i) - f(x_i)| < r_i$ for all $n \geq n_i$. Thus $f_n - f \in V(x, r_i)$ for all $n \geq n_i$. Letting $N = \max\{n_1, \ldots, n_m\}$ we see that for all $n \geq N$, $f_n - f \in \bigcap_{i=1}^{m} V(x_i, r_i)$. Thus $f_n - f \to 0$, hence $f_n \to f$.

We now examine an example to see how the weak-* topology relates to other ways of topologizing a dual space.

2.2.10 Example. Recall that $c_0^* = \ell_1$, and $\ell_1^* = \ell_\infty$. Since $\ell_1$ is a dual space, we can equip it with the weak-* topology. However, we may also equip it with the operator norm inherited from $c_0$, or by the weak topology induced by its dual $\ell_\infty$.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\ell_1$. We consider what it means for $x_n \to 0$ in these three different topologies:

- $(x_n)_{n \in \mathbb{N}} \to 0$ w.r.t. $\|\cdot\|_1$ iff $\sum_{j=1}^{\infty}|(x_n)_j| \to 0$ as $n \to \infty$.
- $(x_n)_{n \in \mathbb{N}} \to 0$ w.r.t. the weak topology iff for each $y \in \ell_\infty$:
  \[ \langle x_n, y \rangle = \sum_{j=1}^{\infty}(x_n)_j y_j \to 0 \text{ as } n \to \infty. \]
- $(x_n)_{n \in \mathbb{N}} \to 0$ w.r.t. the weak-* topology iff for each $y \in c_0$:
  \[ \langle x_n, y \rangle = \sum_{j=1}^{\infty}(x_n)_j \overline{y_j} \to 0 \text{ as } n \to \infty. \]

So we see that $\ell_1$-convergence $\implies$ weak convergence $\implies$ weak-* convergence. We now consider whether the reverse implications are true for this example.

- Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\ell_1$ defined by $(x_n)_j = -1$ if $j = n$, $(x_n)_j = -1$ if $j = n + 1$, and $(x_n)_j = 0$ otherwise. Let $y \in c_0$, i.e. $y_j \to 0$. Note that $\langle x_n, y \rangle = (-1)^n(y_{n+1} - y_n)$. Since $y_j \to 0$, we have $\overline{y_j} \to 0$, hence $\langle x_n, y \rangle \to 0$. Thus we have weak-* convergence of $(x_n)_{n \in \mathbb{N}}$ to 0.

Now, let $z \in \ell_\infty$ be defined by $z_j = (-1)^j$. Then $\langle x_n, z \rangle = (-1)^n \cdot 2$ for all $n \in \mathbb{N}$. Hence $\langle x_n, z \rangle \not\to 0$. Thus $(x_n)$ does not weakly converge to 0. This shows that weak-* convergence does not imply weak convergence for $\ell_1$. 

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- It turns out that weak convergence in $\ell_1$ does imply strong convergence, although this is not true for $\ell_p$ with $1 < p < \infty$. The proof is nontrivial, so see Conway’s *A Course in Functional Analysis* for details (Rudin simply left it as an exercise).

Since the weak-* topology is so coarse, it has a nice compactness property: the closed unit ball of $X^*$ is compact in the weak-* topology.

2.2.11 Theorem. (Banach-Alaoglu Theorem) Let $X$ be a topological vector space, $V \in \mathcal{U}$, and $K = \{ f \in X^* : |f(x)| \leq 1 \ \forall x \in V \}$. Then $K$ is weak-* compact.

Proof. Note that $V$ is absorbing since it is a neighborhood of 0, hence for each $x \in X$ there exists some $\beta(x) > 0$ such that $x \in \beta(x)V$, i.e. $\frac{1}{\beta(x)}x \in V$.

For each $x \in X$, let $D_x = \{\alpha \in \mathbb{F} : |\alpha| \leq \beta(x)\}$, and define $P = \prod_{x \in X} D_x$ with the product topology. Note that each $D_x$ is closed and bounded in $\mathbb{F}$, hence compact. So we have that $P$ is compact by Tychonoff’s theorem.

Note that every element in $P$ is actually a function $f : X \to \mathbb{F}$ with the property that $|f(x)| \leq \beta(x)$ (these functions in $P$ need not be linear). Since every $f \in K$ has this property, we see that $K \subset X^* \cap P$.

So $K$ inherits two topologies: one from the weak-* topology on $X^*$, and one from the product topology on $P$. To proceed, we need to show that these topologies are actually the same.

2.2.12 Lemma. The weak-* topology and product topology induced on $K$ coincide.

Proof. Let $f_0 \in K$. Choose any $x_i \in X$ for $1 \leq i \leq n$, and choose $\delta > 0$. Define the sets:

$$W_1 = \{ f \in X^* : |f(x_i) - f_0(x_i)| < \delta \ \text{for} \ 1 \leq i \leq n \}$$

$$W_2 = \{ f \in P : |f(x_i) - f_0(x_i)| < \delta \ \text{for} \ 1 \leq i \leq n \}.$$

Let $n, x_i, \text{ and } \delta$ range over all possible values. Then $W_1$ forms a local base at $f_0$ for the weak-* topology of $X^*$, and $W_2$ forms a local base at $f_0$ for the product topology of $P$. Since $K \subset X^* \cap P$, we see that $W_1 \cap K = W_2 \cap K$, so both topologies that $K$ inherits coincide.

If we could now show that $K$ is a closed subset of $P$, then that would mean $K$ is compact with respect to the product topology. But since the product topology and weak-* topology on $K$ are actually equivalent, we would have that $K$ is weak-* compact.

2.2.13 Lemma. $K$ is a closed subset of $P$ in the product topology.

Proof. Let $f_0$ in the product-closure of $K$ (we want to show $f_0 \in K$). For any $\alpha, \beta \in \mathbb{F}$, $x, y \in X$, and $\epsilon > 0$ we may define a neighborhood of 0:

$$S = \{ f \in P : |f(x) - f_0(x)| < \epsilon, |f(y) - f_0(y)| < \epsilon, \ \text{and} \ |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon \}.$$
By $f_0$ being in $\mathcal{K}$, there is an $f \in K \cap S$, i.e. $|f(x) - f_0(x)| < \epsilon$, $|f(y) - f_0(y)| < \epsilon$, and $|f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon$. From these inequalities (and linearity of $f$) we get:

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)|$$

$$= |f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) + \alpha f(x) + \beta f(x) - \alpha f(x) - \beta f(x)|$$

$$\leq |f_0(\alpha x + \beta y) - f(\alpha x + \beta y)| + |\alpha||f_0(x) - f(x)| + |\beta||f_0(y) - f(y)|$$

$$\leq \epsilon + |\alpha|\epsilon + |\beta|\epsilon.$$  

Since this holds for any $\epsilon > 0$ and any $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$, we have that $f_0$ is linear.

Lastly, for any $x \in V$ and $\epsilon > 0$, define $S' = \{f \in P : |f(x) - f_0(x)| < \epsilon\}$. Since $S'$ is a neighborhood of $f_0$, there exists some $f \in K \cap S'$, i.e. $|f(x) - f_0(x)| < \epsilon$. Thus:

$$|f_0(x)| = |f_0(x) - f(x) + f(x)| \leq |f_0(x) - f(x)| + |f(x)|.$$  

Since $f \in L$, we know $|f(x)| \leq 1$, hence $|f_0(x)| \leq \epsilon + 1$. Since $\epsilon > 0$ was arbitrary, we conclude that $|f_0(x)| \leq 1$. Thus $f_0 \in K$, so $K$ is closed with respect to the product topology from $P$. \qed

This completes the proof. \qed