Functional Analysis, Math 7320
Lecture Notes from November 22, 2016

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Last time:

1. Weak topology
2. sequential compactness
3. boundedness
4. $\overline{E} \subset \overline{E}^w$

4.1.16 Theorem. *If E is a convex subset of a topological vector space then $\overline{E} = \overline{E}^w$.*

Proof. One direction was proved last time. For the other direction we will use the following version of Hahn-Banach: if A, B are convex and disjoint with A compact and B closed, then there exists a linear functional f such that $\sup_{a\in A} \Re f(a) < \inf_{b\in B} \Re f(b)$. Choose $x_0 \notin \overline{E}$. Since $\{x_0\}$ is compact, there exists f as above and $\beta \in \mathbb{R}$ such that $\Re f(x_0) < \beta < \inf_{x\in\overline{E}} \Re f(x)$. Then $\{x \colon \Re f(x) < \beta\}$ is a weak neighborhood of x_0 disjoint from E. So $x_0 \notin \overline{E}^w$. □

4.1.17 Corollary. *If E is a convex subset of a metrizable topological vector subspace X and x_n is a sequence in E which converges weakly to x (hence $x \in \overline{E}^w$), then there exists a sequence $\{y_n\}$ in E such that $y_n \to x$ in the topology of X.*

Proof. This follows immediately since $x \in \overline{E}^w = \overline{E}$. □

4.1.18 Corollary. *If E is a convex subset of a locally convex topological vector space, then:*

1. *E is τ-closed if and only if E is weakly closed.*
2. *E is τ-dense if and only if E is weakly dense.*

Proof. This is obvious from the theorem. □

5 The weak-* topology

We know X^* is a vector space. If X is normed, we can make X^* a Banach space by equipping it with the operator norm. But if X is merely a topological vector space, what additional structure can we give X^*? We can give X^* the weak topology, which is the initial topology on X^* induced by X^{**}. But in many applications it is much more useful to weaken this topology.

Towards this end, observe that we can embed X into X^{**} via the canonical map $i\colon X \to X^{**}$ given by $i(x)(f) \equiv F_x(f) = f(x)$ for all $f \in X^*$. We claim that the set $\{F_x\}_{x\in X}$ separates points. For if $f, g \in X^*$ and $F_x(f) = F_x(g)$ for all $x \in X$, then $f(x) = g(x)$ for all $x \in X$ which means $f = g$. We call the initial topology on X^* induced by $\{F_x\}_{x\in X}$ the weak-* topology on X^*. A local subbasis for this topology is $\{V(x,r)\}_{x\in X, r>0}$ where $V(x,r) = \{f \in X^*\colon |f(x)| < r\}$.

Note that a net $\{f_\alpha\}$ in X^* converges weak-* to $f \in X^*$ if and only if $f_\alpha(x) \to f(x)$ for all $x \in X$.

5.0.1 Example. Recall that $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$. Then ℓ_1 has three topologies: the norm, weak, and weak-* topologies. Suppose $\{x_n\}$ is a sequence in ℓ_1. Then $x_n \to 0$ in norm means that $\sum_{j=1}^\infty |(x_n)_j| \to 0$ as $n \to \infty$. Next, $x_n \to 0$ weakly means that $\langle x_n, y\rangle = \sum_{j=1}^\infty (x_n)_j \overline{y_j} \to \infty$ as $n \to \infty$ for all $y \in \ell_\infty$. Finally, $x_n \to 0$ weak-* means that $\langle x_n, y\rangle \to 0$ for all $y \in c_0$. Thus ℓ_1 convergence implies weak convergence implies weak-* convergence.

5.0.2 Theorem (Alaoglu)**.** *If (X,τ) is a topological vector space, $V \in \mathcal{O}(0)$, then the closed unit ball K of X^* is weak-* compact.*

Before we prove this, for each $x \in X$ define a set $D_x = \{k \in \mathbb{K}\colon |k| \le \|x\|\} = \overline{B}_{\|x\|}(0) \subset \mathbb{K}$. Then define $D = \prod_{x\in X} D_x$ and give D the product topology. D is then the set of all functions $\varphi\colon X \to \mathbb{K}$ such that $|\varphi(x)| \le \|x\|$ for every $x \in X$. Note that for any $f \in B^*$ we have $|f(x)| \le \|f\|_{X^*}\|x\| \le \|x\|$ so that $B^* \subset D$. We need a lemma:

5.0.3 Lemma. *The relative topology $\tau^p_{B^*}$ that B^* inherits from the product topology on D and the relative topology $\tau^*_{B^*}$ that B^* inherits from the weak-* topology on X^* coincide.*

Proof. We prove this lemma by characterizing the convergence of nets in both topologies. First suppose that $\{f_\alpha\}_{\alpha\in A}$ is a net in B^* which converges to $f \in B^*$ with respect to $\tau^p_{B^*}$. We claim that $f_\alpha(x) \to f(x)$ for all $x \in X$. To see this, choose $x_0 \in X$ and $\epsilon > 0$. The set $\{k \in \mathbb{K}\colon |f(x_0) - k| < \epsilon\}$ is open in $\mathbb{K}$ and so $S_{x_0} = \{k \in \mathbb{K}\colon |f(x_0) - k| < \epsilon\} \cap D_{x_0}$ is open in D_{x_0}. Therefore the set $S = \prod_{x\in X} S_x$, where $S_x = D_x$ whenever $x \ne x_0$, is open in D. So by assumption there exists $\alpha_0 \in A$ such that $\alpha \ge \alpha_0$ implies $f_\alpha \in S$. But this means in particular that for $\alpha \ge \alpha_0$ we have $f_\alpha(x_0) \in S_{x_0}$ which yields $|f(x_0) - f_\alpha(x_0)| < \epsilon$. Since ϵ was arbitrary this proves that $f_\alpha \to f$ pointwise.

Conversely suppose that $\{f_\alpha\}_{\alpha\in A}$ is a net in B_{X^*} which converges to $f \in B^*$ pointwise. We claim that $f_\alpha \to f$ with respect to $\tau^p_{B^*}$. To see this, choose a basis element U for $(B^*, \tau^p_{B^*})$ containing f. Write $U = \prod_{x\in X} U_x$ where U_x is open in D_x and all but finitely many of the sets U_x are equal to D_x. Let $\{x_i\}_{i=1}^n$ be the indices for which $U_{x_i} \ne D_{x_i}$. So U_{x_i} is open in D_{x_i} and contains the point $f(x_i)$. Write $U_{x_i} = D_{x_i} \cap V_i$ where V_i is open in $\mathbb{K}$. Notice that $f(x_i) \in V_i$. Then by assumption there exist indices $\{\alpha_i\}_{i=1}^n$ such that for each i we have $\alpha \ge \alpha_i$ implies $f_\alpha(x_i) \in V_i$. Since $f_\alpha(x_i) \in D_{x_i}$ also (since $f_\alpha \in B^*$) this means that $f_\alpha \in U_{x_i}$ for $\alpha \ge \alpha_i$.

Now let α_0 be an upper bound for $\alpha_1, \ldots, \alpha_n$ in A. Then if $\alpha \geq \alpha_0$ we have $f_\alpha(x_i) \in U_{x_i}$ for $i = 1, \ldots, n$. Hence $f_\alpha \in U$ for $\alpha \geq \alpha_0$ which shows that $f_\alpha \to f$ in $\tau^p_{B^*}$.

We have proved that a net converges to a point f in $(B^*, \tau^p_{B^*})$ if and only if the net converges to f pointwise. We now show that the same is true for nets in $(B^*, \tau^*_{B^*})$.

So suppose that $\{f_\alpha\}_{\alpha \in A}$ is a net in B^* which converges to $f \in B^*$ with respect to $(B^*, \tau^*_{B^*})$. Choose $x \in X$ and $\epsilon > 0$. Since $\{\varphi \in X^* \colon |\widehat{x}(\varphi) - \widehat{x}(f)| < \epsilon\}$ is the inverse image of the open set $B_\epsilon(\widehat{x}(f)) \subset \mathbb{K}$ under the map $\widehat{x} \in X^{**}$, it is weak-* open in X^*. Hence $S = \{\varphi \in X^* \colon |\widehat{x}(\varphi) - \widehat{x}(f)| < \epsilon\} \cap B^* \in \tau^*_{B^*}$ and contains f. So by assumption there exists $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $f_\alpha \in S$. But this says that $|\widehat{x}(f_\alpha) - \widehat{x}(f)| < \epsilon$, or equivalently, $|f_\alpha(x) - f(x)| < \epsilon$, for $\alpha \geq \alpha_0$. Since x and ϵ were arbitrary this shows that $f_\alpha \to f$ pointwise.

Conversely, suppose that the net $\{f_\alpha\}_{\alpha \in A}$ in B^* converges to $f \in B^*$ pointwise. Then $f_\alpha(x) \to f(x)$ for all $x \in X$, which is equivalent to saying $\widehat{x}(f_\alpha) \to \widehat{x}(f)$ for all $x \in X$. This in turn means that $f_\alpha \to f$ in (X^*, τ_{w^*}). To show that $f_\alpha \to f$ in $(B^*, \tau^*_{B^*})$ choose any set $U \in \tau^*_{B^*}$ containing f and write $U = B^* \cap V$ where $V \in \tau_{w^*}$. Then there exists $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $f_\alpha \in V$. But $f_\alpha \in B^*$ so that $f_\alpha \in U$ for $\alpha \geq \alpha_0$. This proves that $f_\alpha \to f$ in $(B^*, \tau^*_{B^*})$.

We have proved that a net converges to a point f in $(B^*, \tau^*_{B^*})$ if and only if the net converges to f pointwise. We conclude that a net in B^* converges to a point $f \in B^*$ with respect to $\tau^p_{B^*}$ if and only if the net converges to f with respect to $\tau^*_{B^*}$. Since any topology is characterized by the behavior of its convergent nets, this shows that $\tau^p_{B^*} = \tau^*_{B^*}$. $\square$

Proof of Alaoglu's Theorem. V is absorbing so each $x \in X$ has $\beta(x) > 0$ such that $x \in \beta(x)V$. So for $x \in X$ and $f \in K$ we have $|f(x)| = \beta(x) \cdot |f(\frac{x}{\beta(x)})| \leq \beta(x)$ since $\frac{x}{\beta(x)} \in V$. In the above lemma replace D_x with $B_\beta(x)(0)$ and the same result follows.

To complete the proof we need only show that K is closed in D with the product topology. So choose f_0 in the product topology closure of K. Choose a typical neighborhood S of f_0. Write $S = \{f \in D \colon |f(x) - f_0(x)| < \epsilon, |f(y) - f_0(y)| < \epsilon, |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \epsilon\}$. Then there exists $f \in K$ with $f \in S$. But then

$$|f_0(\alpha x + \beta y) - \alpha f(x) - \beta f(y) + \alpha f(x) + \beta f(y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\epsilon$$

which implies that f_0 is linear.

To be continued... $\square$