4.2.5 Theorem. If $X$ is a locally-convex topological vector space and $E \subseteq X$ is convex, then the weak closure $\overline{E}^w$ of $E$ is equal to its original closure $\overline{E}$.

Proof. $\overline{E}^w$ is weakly closed, which implies that it is originally closed, which in turn implies that $\overline{E} \subseteq \overline{E}^w$. Conversely, let $x_0 \in X$ such that $x_0 \notin \overline{E}$. We use the following result of Hahn-Banach:

If $A$ and $B$ are disjoint, nonempty, convex subsets of a locally-convex topological vector space; $A$ is compact; and $B$ is closed, then there is a continuous linear functional $f$ on $X$ such that $\sup \Re f(A) < \inf \Re f(B)$.

This implies that there is a $\beta \in \mathbb{R}$ such that $\Re f(x_0) < \beta < \inf \Re f(\overline{E})$. Therefore, $\{x \in X : \Re f(x) < \beta\}$ does not intersect $E$ and is a weak open neighborhood of $x_0$, which implies that $x_0 \notin \overline{E}^w$, which in turn implies that $\overline{E}^w \subseteq \overline{E}$ after taking complements. \hfill \Box

4.2.6 Corollary. If $X$ is a metrizable, convex, topological vector space; $E \subseteq X$ is convex; and $(x_n)_{n \in \mathbb{N}}$ is a sequence in $E$ that converges weakly to $x \in X$, then there is a sequence $(y_n)_{n \in \mathbb{N}}$ in $E$ that converges originally to $x$.

Proof. Let $H$ be the convex hull of the set of all $x_n$ and let $K$ be the weak closure of $H$. Then $x \in K$. By the above theorem, $x$ is in the original closure of $H$. Since the original topology is metrizable, there is a sequence $(y_n)_{n \in \mathbb{N}}$ in $H$ that converges originally to $x$. \hfill \Box

4.2.7 Corollary. For a convex subset $E$ of a locally-convex topological vector space:

(1) $E$ is $\tau$-closed if and only if $E$ is $\tau_w$-closed.

(2) $E$ is $\tau$-dense if and only if $E$ is $\tau_w$-dense.

Proof.

(1) The result follows from $\overline{E} = \overline{E}^w$.

(2) The result follows from (1): $\overline{E} = X$ if and only if $\overline{E}^w = X$. \hfill \Box
4.3 The Weak-* Topology on $X^*$

So far, we have considered $X^*$ as a vector space. If $X$ is normed, then we can equip $X^*$ with the operator norm to turn it into a Banach space. However, what if $X$ is a topological vector space? We will use a space of linear functionals on $X$ to define a topology on $X^*$:

Consider $x \mapsto F_x$ on $X$, where $F_x(f) = f(x)$ for each $f \in X^*$. Then each $F_x$ is linear because

$$F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g),$$

and $\{F_x\}_{x \in X}$ separates points in $X^*$ because if $f(x) = g(x)$ for each $x \in X$, then $f = g$. Hence, $\{F_x\}_{x \in X}$ induces a topology on $X^*$. This topology is called the weak-* topology.

4.3.8 Definition. The **weak-* topology** on $X^*$ is generated by the local subbase $\{v(x, r)\}_{x \in X, r > 0}$, where $v(x, r) = \{f \in X^* : |f(x)| < r\}$. Hence, weak-* convergence of a sequence $(f_n)_{n \in \mathbb{N}}$ to $f$ means that for each $x \in X$, $\lim_{n \to \infty} f_n(x) = f(x)$.

4.3.9 Example. Recall that $c_0^* = \ell_1$ and $\ell_1^* = \ell^\infty$. Since $\ell_1$ is a dual space, we can equip it with the weak-* topology.

- If $(x_n)_{n \in \mathbb{N}} \overset{\|\cdot\|_1}{\to} 0$, then $\sum_{j=1}^{\infty} |(x_n)_j| \overset{n \to \infty}{\to} 0$.
- $(x_n)_{n \in \mathbb{N}} \overset{w}{\to} 0$ if and only if for each $y \in \ell^\infty$, $\langle x_n, y \rangle = \sum_{j=1}^{\infty} (x_n)_j y_j \overset{n \to \infty}{\to} 0$.
- $(x_n)_{n \in \mathbb{N}} \overset{w^*}{\to} 0$ if and only if for each $y \in c_0$, $\langle x_n, y \rangle \overset{n \to \infty}{\to} 0$.

So $\ell_1$ convergence implies weak convergence, which in turn implies weak-* convergence. Since the weak-* topology is coarse, it has a nice compactness property:

4.3.10 Theorem. Let $(X, \tau)$ be a topological vector space, let $V \in \mathcal{U}$, and let

$$K = \{f \in X^* : |f(x)| \leq 1 \text{ for all } x \in V\}.$$  

Then $K$ is weak-* compact.

Proof. Since $V$ is absorbing, for each $x \in X$, there is a $\beta(x) > 0$ such that $x \in \beta(x) V$. Thus for $x \in X$ and $f \in K$,

$$|f(x)| = \beta(x) |f \left( \frac{x}{\beta(x)} \right)| \leq \beta(x).$$

Let $D_x = \{\alpha \in \mathbb{F} : |\alpha| \leq \beta(x)\}$ and let $P = \prod_{x \in X} D_x$ be equipped with the product topology. Since each $D_x$ is compact, Tychonoff’s theorem implies that $P$ is compact. Then every element of $P$ is a function $f : X \to \mathbb{F}$ such that $|f(x)| \leq \beta(x)$. This implies that every element of $K$ is in $P$, which in turn implies that $K \subseteq X^* \cap P$.

We interject a lemma:

4.3.11 Lemma. The weak-* topology and the product topology induced on $K$ coincide.
Proof. Let \( f_0 \in K \), let \( x_1, \ldots, x_n \in X \), let \( \delta > 0 \), let

\[
W_1 = \{ f \in X^* : |f(x_j) - f_0(x_j)| < \delta \text{ for } 1 \leq j \leq n \}, \text{ and let}
\]

\[
W_2 = \{ f \in P : |f(x_j) - f_0(x_j)| < \delta \text{ for } 1 \leq j \leq n \}.
\]

Then as \( n, x_i \), and \( \delta \) range over all possible values, the resulting sets \( W_1 \) and \( W_2 \) form local bases for the weak-* topology and the product topology at \( f_0 \) of \( X^* \) and \( P \), respectively. However, since \( K \subseteq X^* \cap P \), we have that \( W_1 \cap K = W_2 \cap K \), which implies that both topologies restricted to \( K \) coincide.

We interject another lemma:

4.3.12 Lemma. \( K \) is a closed subset of \( P \) with respect to the product topology.

Proof. Let \( f_0 \) be in the closure of \( K \) with respect to the product topology, let \( x, y \in X \), let \( \alpha, \beta \in \mathbb{F} \), and let \( \varepsilon > 0 \). Then

\[
N = \{ f \in P : |f(x) - f_0(x)| < \varepsilon, |f(y) - f_0(y)| < \varepsilon, \text{ and } |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)| < \varepsilon \}
\]

is a neighborhood of \( f_0 \) in the product topology. Therefore, there is an \( f \in K \) such that \( f \in N \). Since \( f \) is linear, we have that

\[
\begin{align*}
f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) &= f_0(\alpha x + \beta y) - f(\alpha x + \beta y) \\
&\quad + f(\alpha x + \beta y) - f_0(\alpha x + \beta y) \\
&= (f_0 - f)(\alpha x + \beta y) + \alpha f(x) + \beta f(y) - \alpha f_0(x) - \beta f_0(y) \\
&= (f_0 - f)(\alpha x + \beta y) + \alpha (f - f_0)(x) + \beta (f - f_0)(y),
\end{align*}
\]

which implies that

\[
|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\varepsilon.
\]

Hence, \( f_0 \) is linear. If \( x \in V \) and \( \varepsilon > 0 \), the same argument shows that there is an \( f \in K \) such that \( |f(x) - f_0(x)| < \varepsilon \). Since \( |f(x)| \leq 1 \), by the definition of \( K \), it follows that \( |f_0(x)| \leq 1 \). As a result, \( f_0 \in K \).

This proves the theorem.