Functional Analysis, Math 7320
Lecture Notes from November 29, 2016

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Last time we were in the middle of the proof of Alaoglu's Theorem. We now resume it.

Proof of Alaoglu's Theorem (continued). We showed that there exists $f \in K \cap S$. Hence $|f(x) - f_0(x)| < \epsilon$ which implies that $|f_0(x)| < 1 + \epsilon$. Since $x \in K$ and $\epsilon > 0$ were arbitrary this shows that $\|f_0\|_{X^*} \leq 1$ and hence $f_0 \in K$. So K is closed in D with the product topology. But D is compact and so K is compact in the product topology. By the lemma above this means that K is compact in the weak-* topology and the proof is complete. ☐

If X is separable we can say a bit more.

5.1.16 Theorem. *If X is separable then $(B^*, \tau^*_{B^*})$ is metrizable.*

Proof. Choose a dense subset $\{x_n\}_{n=1}^{\infty}$ of X. We first show that the functionals $\{\widehat{x_n}\}_{n=1}^{\infty}$ separate points in X^*. Indeed, suppose that $f, g \in X^*$ and that $\widehat{x_n}(f) = \widehat{x_n}(g)$ for all $n \in \mathbb{N}$. Then $f(x_n) = g(x_n)$ for all n, and hence f and g agree on the dense subset $\{x_n\}_{n=1}^{\infty} \subset X$. Since f and g are continuous, this means that $f = g$.

For $n \in \mathbb{N}$ define $y_n = \|x_n\|^{-1} x_n$ if $x_n \neq 0$ and $y_n = 0$ if $x_n = 0$. Recall that $\|\widehat{x_n}\|_{X^{**}} = \|x_n\|$. Now define a function $d\colon B^* \times B^* \to \mathbb{R}$ by $d(f, g) = \sum_{n=1}^{\infty} 2^{-n} |\widehat{y_n}(f) - \widehat{y_n}(g)|$. It is trivial that d is a metric (the proof of positive-definiteness uses the fact that the $\widehat{x_n}$'s separate points). Also the sum in the definition of d converges uniformly on $B^* \times B^*$ (because of the linearity of $\widehat{x_n}$) so that d is a uniform limit of continuous functions on the compact space $(B^*, \tau^*_{B^*}) \times (B^*, \tau^*_{B^*})$, and so d is continuous with respect to the product topology on $(B^*, \tau^*_{B^*}) \times (B^*, \tau^*_{B^*})$. Let τ_d be the topology on B^* induced by d.

To show that $\tau_d = \tau^*_{B^*}$, choose $B_r(f) \in \tau_d$. Then $B_r(f) = \{g \in B^* \colon d(f, g) < r\}$ is the inverse image of the open set $(-r, r)$ under the weak-* continuous map $d(f, \cdot)$, and so is weak-* open. Hence $\tau_d \subset \tau^*_{B^*}$. Conversely, choose any $\tau^*_{B^*}$-closed set F. Now any τ_d-open cover $\{U_\alpha\}$ of F is a $\tau^*_{B^*}$-open cover of F. Since F is closed in the compact and Hausdorff space $(B^*, \tau^*_{B^*})$ it is compact there, and so a finite subcover $\{U_n\}_{n=1}^{k}$ covers F. Thus F is τ_d-compact and so F is τ_d-closed. Thus every $\tau^*_{B^*}$-closed set is also τ_d-closed, which implies that $\tau^*_{B^*} \subset \tau_d$. We have shown that $\tau^*_{B^*} = \tau_d$ and hence that $(B^*, \tau^*_{B^*})$ is metrizable. ☐

5.1.17 Corollary. *If $V \in \mathcal{O}(x)$ in a topological vector space and $\{f_n\}$ is a sequence in the closed ball K of X^*, there exists an accumulation point $f \in K$ of $\{f_n\}$.*

Proof. Since K is weak-* compact and metrizable it is weak-* sequentially compact. ☐

6 The Krein-Milman Theorem

In this section we prove that a set is the convex hull of its extreme points.

6.0.1 Definition. Let E be a subset of a vector space V. The convex hull of E, $\mathrm{co}(E)$, is the intersection of all convex sets containing E.

6.0.2 Proposition. $\mathrm{co}(E) = \{\sum_{j=1}^{n} \lambda_j x_j \colon \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, x_j \in E\}$.

Proof. Let $S = \{\sum_{j=1}^{n} \lambda_j x_j \colon \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, x_j \in E\}$. First we show that S is convex. So choose $\lambda \in [0,1]$ and two elements $\sum_{j=1}^{n} \lambda_j x_j$ and $\sum_{j=1}^{m} \lambda'_j x'_j$ of S. Then

$$\begin{aligned} \lambda \sum_{j=1}^{n} \lambda_j x_j + (1-\lambda) \sum_{j=1}^{m} \lambda'_j x'_j &= \sum_{j=1}^{n} \lambda\lambda_j x_j + (1-\lambda)\lambda'_j x'_j \\ &= \sum_{j=1}^{m+n} \lambda''_j x''_j \end{aligned}$$

where

$$\lambda''_j = \begin{cases} \lambda\lambda_j & \text{if } 1 \leq j \leq n \\ (1-\lambda)\lambda'_{j-n} & \text{if } j > n \end{cases}$$

$$x''_j = \begin{cases} x_j & \text{if } 1 \leq j \leq n \\ x'_{j-n} & \text{if } j > n \end{cases}.$$

Then $x''_j \in E$ and $\lambda_j \geq 0$ for all j and

$$\sum_{j=1}^{m+n} \lambda''_j = \sum_{j=1}^{n} \lambda\lambda_j + \sum_{j=1}^{m} \lambda\lambda'_j = \lambda + (1-\lambda) = 1$$

which proves that $\lambda \sum_{j=1}^{n} \lambda_j x_j + (1-\lambda) \sum_{j=1}^{m} \lambda'_j x'_j \in S$. So S is convex as claimed.

Next, taking $n = 1$ in the definition of S shows that $E \subset \mathrm{co}(E)$. So S is a convex set containing E and thus $\mathrm{co}(E) \subset S$.

Finally, choose any convex set $D \supset E$. Then any convex combination of elements of E is also in D. Hence $S \subset D$ which implies that $S \subset \mathrm{co}(E)$. □

6.0.3 Definition. Suppose E is a subset of a topological vector space. The *closed convex hull* of E, $\overline{\mathrm{co}}(E)$, is the closure of $\mathrm{co}(E)$.

6.0.4 Definition. Suppose E is a subset of a topological vector space X. E is *totally bounded* if for every $U \in \mathcal{O}(0)$ there exists a finite set $F \subset X$ such that $E \subset F + V$.