Dual Spaces, Riesz Representation Theorem, and Summability
Lecture Notes from August 30, 2022
taken by An Vu

Last Time

- Direct sum
- Orthogonal projections and orthogonal spaces

Warm up

- Recall the corollary from last time: Suppose $E$ is a subspace of a Hilbert space $\mathcal{H}$. Then $E$ is closed if and only if $(E^\perp)^\perp = E$. The proof for the preceding theorem was quite lengthy, but a shorter proof of the corollary can be found in Rudin: Since $E$ is closed, the direct sums

$$\bar{E} \oplus \bar{E}^\perp = \mathcal{H}$$

and

$$\bar{E}^\perp \oplus (\bar{E}^\perp)^\perp = \mathcal{H}$$

are unique. Comparing the two identities, we identify that $\bar{E}$ must be $(\bar{E}^\perp)^\perp$.

- Recall the steps to prove completeness of $l^2 \equiv l^2(\mathbb{N})$:

  1. Consider any Cauchy sequence $(x^n)_{n \in \mathbb{N}}$ in $l^2$. Using norm inequality

     $$|x^n_j - x^m_j| \leq \|x^n - x^m\|,$$

     we get that, for each fixed $j$, $(x^n_j)_{n \in \mathbb{N}}$ forms a Cauchy sequence in $\mathbb{C}$, hence converges to $x^n_j \to x_j$ by completeness.

  2. Using that Cauchy sequences are bounded, show $(x_j)_{j \in \mathbb{N}} \in l^2(\mathbb{N})$.

  3. Show sequence $(x^n)_{n \in \mathbb{N}}$ actually converges to $x$ with respect to the norm on $l^2$, which can be shown by using triangle inequality and sup argument.
Dual Spaces

Consider the space of bounded linear functionals on $\mathcal{H}$.

1.14 Definition. The dual space $V'$ of a normed space $V$ is given by all linear maps

$$\Lambda : V \rightarrow \mathbb{C}, \quad \sup_{\|x\| \leq 1} |\Lambda x| < \infty.$$ 

We equip $V'$ with the norm

$$\|\Lambda\| = \sup_{\|x\| \leq 1} |\Lambda x|.$$ 

To see that $\|\Lambda\|$ is indeed a norm on $\mathcal{H}$, it is enough to show that it is positive definite, i.e., we want to show that

$$\|\Lambda\| = 0 \iff \Lambda = 0.$$ 

Suppose that $\|\Lambda\| = 0$, or $\sup_{\|x\| \leq 1} |\Lambda x| = 0$, which implies $|\Lambda x| \leq 0$ for all $x \in \mathcal{H}$, so it must be true that $\Lambda x = 0$ for all $x$. Thus, $\Lambda = 0$.

Conversely, let $\Lambda = 0$, then $\sup_{\|x\| \leq 1} |\Lambda x| = 0 = \|\Lambda\|$.

Riesz Representation Theorem

1.15 Theorem. Let $\mathcal{H}$ be a complex Hilbert space. Then the map $\phi : \mathcal{H} \rightarrow \mathcal{H}'$ given by

$$\phi(x) (y) = \langle y, x \rangle$$

is a conjugate linear isometric bijection.

In particular, if $\Lambda$ is a bounded linear functional on $\mathcal{H}$, then there is an $x \in \mathcal{H}$ so that for each $y \in \mathcal{H}$,

$$\Lambda y = \langle y, x \rangle,$$

and

$$\|\Lambda\| = \|x\|.$$ 

Proof. (This proof may be found in Rudin, Chapter 12, page 308. I typed it here for convenience and added a few notes for my own understanding).

We first want to show $\|\Lambda\| = \|x\|$ for $x \in \mathcal{H}$. By Cauchy-Schwarz inequality, we have that

$$|\Lambda y| := |\langle y, x \rangle| \leq \|y\| \|x\|,$$

which implies $\|\Lambda\| \leq \|x\|$. Conversely, $\|x\|^2 = \langle x, x \rangle = |\Lambda x| \leq \|\Lambda\| \|x\|$, so $\|x\| \leq \|\Lambda\|$.

Next, we want to show that every $\Lambda \in \mathcal{H}$ has the form $\Lambda y = \langle y, x \rangle$ for $y \in \mathcal{H}$. If $\Lambda = 0$, then we take $x = 0$. If $\Lambda$ is non-zero, we let $\mathcal{N}(\Lambda) := \{y \in \mathcal{H} : \Lambda y = 0\}$ be the null space of $\Lambda$. Since $\mathcal{H} = \mathcal{N}(\Lambda) \oplus (\mathcal{N}(\Lambda))^\perp$, there exists a $z \in (\mathcal{N}(\Lambda))^\perp, z \neq 0$, and

$$(\Lambda y)z - (\Lambda z)y \in \mathcal{N}(\Lambda)$$

for all $y \in \mathcal{H}$, which implies

$$(\Lambda y)\langle z, z \rangle - (\Lambda z)\langle y, z \rangle = 0$$

$$\iff \Lambda y = \frac{1}{\langle z, z \rangle} \Lambda z \langle y, z \rangle = \langle y, \tilde{\Lambda} z \rangle (\frac{z}{\langle z, z \rangle}).$$

Hence, for any $y \in \mathcal{H}$, $\Lambda y = \langle y, x \rangle$ with $x = (\tilde{\Lambda} z) (\frac{z}{\langle z, z \rangle}) \in \mathcal{H}$. 


Summability

In this section, we introduce a more general version of direct sum.

First, let us recall the definition of summability.

1.16 Definition. Let $V$ be a normed space, and $(x_j)_{j \in J}$ a family of elements in $V$, so $x : J \to V, j \mapsto x_j$ a $V$-valued function.

Let $\mathcal{F}$ be the set of all finite subsets of $J$. Then $(x_j)_{j \in J}$ is called summable if there is a $y \in V$ such that for each $\epsilon > 0$, there is an $F_\epsilon \in \mathcal{F}$ such that for all $F \in \mathcal{F}$ with $F_\epsilon \subset F$,

$$\sum_{j \in F} x_j \in B_\epsilon(y) \equiv \{ x \in V : \| x - y \| < \epsilon \}.$$ 

For example, suppose $V = \mathbb{R}$ and $(x_j)$ are non-negative functions (not necessarily linear), given by $j \mapsto x_j \in \mathbb{R}^+$. If $(x_j)_{j \in J}$ are summable, then

$$\sup \{ \sum_{k=1}^n (x_j)_k : \{ j_1, j_2, \ldots, j_n \} \in J \} < \infty.$$ 

1.17 Remark. We note that this definition is, unlike the convergence of series, invariant reordering $J$ with a bijection since any reordering would create another subset of $J$ that is also finite and in $\mathcal{F}$.

We use the notion of summability to equip a family of Hilbert spaces with a new inner product.

1.18 Lemma. Let $(\mathcal{H}_j)_{j \in J}$ be a family of Hilbert spaces and

$$\mathcal{H} = \{(x_j)_{j \in J} \in \prod_{j \in J} \mathcal{H}_j, \sum_{j \in J} \| x_j \|^2 < \infty \}.$$ 

Then $\mathcal{H}$ is a Hilbert space with the inner product

$$\langle (x_j)_{j \in J}, (y_j)_{j \in J} \rangle = \sum_{j \in J} \langle x_j, y_j \rangle.$$ 

Proof. We first show that $\mathcal{H}$ is subspace of the vector space $\prod_{j \in J} \mathcal{H}_j$:

It is clear to see that $\mathcal{H}$ is closed under scalar multiplication. To see $\mathcal{H}$ is closed under addition, recall the parallelogram law

$$\| a + b \|^2 \leq 2 \| a \|^2 + 2 \| b \|^2$$

for $a, b \in \mathcal{H}$. So for $x = (x_j)_{j \in J}, y = (y_j)_{j \in J} \in \mathcal{H}$,

$$\| x + y \|^2 = \sum_{j \in J} \| x_j + y_j \|^2 \leq 2 \sum_{j \in J} \| x_j \|^2 + 2 \sum_{j \in J} \| y_j \|^2 < \infty,$$

where the last inequality is due to the summability of $x_j, y_j$. Thus, $\mathcal{H}$ is a subspace of $\prod_{j \in J} \mathcal{H}_j$.

Next, for $x, y \in \mathcal{H}$, the polarization identity, $x \pm y \in \mathcal{H}$, and $x \pm iy \in \mathcal{H}$ gives

$$\langle x, y \rangle = \sum_{j \in J} \langle x_j, y_j \rangle.$$
To show that the above identity is an inner product, it is enough to show its positive definiteness. Let \( x \in \mathcal{H} \). Suppose \( \langle x, x \rangle = 0 \), which implies \( \sum_{j \in J} \langle x_j, y_j \rangle = 0 \). Since \( \mathcal{H}_j \) is a Hilbert space for each \( j \), we have that, if \( \langle x_j, x_j \rangle = 0 \), then \( x_j = 0 \), so \( x = 0 \). Conversely, if \( x = 0 \), then \( x_j = 0 \) for all \( j \in J \), and since \( \mathcal{H}_j \) are Hilbert spaces, \( \langle x_j, x_j \rangle = 0 = \sum_{j \in J} \langle x_j, x_j \rangle \).

Finally, we need to show completeness (using the steps provided in the warm-up).

Let \((x^n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \( \mathcal{H} \). Then by norm inequality
\[
\|x_j^n - x_j^k\| \leq \|x^n - x^k\|_{\mathcal{H}},
\]
we have that, for fixed \( j \in J \), \((x^n_j)_{n \in \mathbb{N}}\) is Cauchy, ie. \((x^n_j)_{j \in J}\) converges to \( x_j \) \( \in \mathcal{H}_j \).

For each finite subset \( F \subset J \):
\[
\sum_{j \in F} \|x_j\|_{\mathcal{H}_j}^2 = \lim_{k \to \infty} \sum_{j \in F} \|x_j^k\|_{\mathcal{H}_j}^2 \leq \lim_{k \to \infty} \sum_{j \in J} \|x_j^k\|_{\mathcal{H}_j}^2 = \lim_{k \to \infty} \|x^k\|^2 < \infty,
\]
with the last inequality using boundedness of Cauchy sequences. Hence, \((x_j)_{j \in J} \in \mathcal{H} \).

It remains to show that \( x^n \to x \) as \( n \to \infty \). Let \( \epsilon > 0 \) and choose \( n_0 \in \mathbb{N} \) with \( \|x^n - x^k\| < \epsilon \) for \( n, k \geq n_0 \). Then for each finite \( F \in J \)
\[
\sum_{j \in F} \|x_j - x_j^n\|_{\mathcal{H}}^2 = \lim_{m \to \infty} \sum_{j \in F} \|x_j^m - x_j^n\|_{\mathcal{H}}^2 \leq \lim_{m \to \infty} \sum_{j \in J} \|x_j^m - x_j^n\|_{\mathcal{H}}^2 \leq \epsilon^2,
\]
so \( x^n \to x \), and \( \mathcal{H} \) is complete. \( \square \)

The readers may notice that, if \( \mathcal{H} = \mathbb{R}^n = \prod_n \mathbb{R} \), then the inner product in the above theorem is the same with the usual inner product in \( \mathbb{R}^n \).