Last time

- Characterization of unitaries
- Characterization of isometries (unitaries + a condition)
- Geometric characterization of normal operators
- Analogue of rank-nullity for \( A \in \mathbb{B}(\mathcal{H}) \)
- Characterization of orthogonal projections (proof in today’s class)

Warm up:

1.47 Question. If \( \mathcal{H} \) is a finite-dimensional Hilbert space and \( A : \mathcal{H} \to \mathcal{H} \) a linear map satisfying \( A^*A = \text{id}_{\mathcal{H}} \). Then \( A \) is unitary.

We will see that \( A \) is onto, then (onto + isometry) \( \implies \) unitary. \( A^* \) is onto since \( A^*A = \text{id}_{\mathcal{H}} \) so given any \( x \in \mathcal{H} \), \( A^*Ax = x \) hence there exists \( y = Ax \in \mathcal{H} \) such that \( A^*y = x \). We also know that \( \mathcal{N}(A) = \mathcal{R}(A)^\perp = \{0\} \) and so \( A \) is one-one. Next, using Rank-Nullity we have that \( \dim \mathcal{H} = \dim \mathcal{N}(A) + \dim \mathcal{R}(A) = \dim \mathcal{R}(A) \), \( \mathcal{H} = \mathcal{R}(A) \) and so \( A \) is onto and hence \( A \) is unitary.

We continue the proof of the theorem from last time that characterized orthogonal projections. Let us recall the theorem

1.48 Theorem. Suppose \( P \in \mathbb{B}(\mathcal{H}) \) be a non-zero projection, i.e., \( P^2 = P \), then TFAE:

1. \( P \) is an orthogonal projection, so \( \mathcal{N}(P) \perp \mathcal{R}(P) \).
2. \( \|P\| = 1 \).
3. \( \langle Px, x \rangle \).
4. \( P = P^* \).
5. \( P \) is normal.
Last time we saw that given any projection (not necessarily orthogonal), we always have the direct sum $H = \mathcal{N}(P) + \mathcal{R}(P)$, $x = Px + (I - P)x$. Here $\mathcal{N}(P) = P^{-1}(\{0\})$ and $\mathcal{R}(P) = (I - P)^{-1}(\{0\})$ (since $x \in (I - P)^{-1}(\{0\}) \iff (I - P)x = 0 \iff Px = x \iff x \in \mathcal{R}(P)$) are both closed subspaces since $P$ is bounded. We prove the following chain of equivalences: (1. $\iff$ 2. and 1. $\implies$ 3. $\implies$ 4. $\implies$ 5. $\implies$ 1.) We already proved that (1. $\implies$ 2.). Now we prove the rest of the equivalences.

**Proof.** (2. $\implies$ 1.) We have $\|P\| = 1$. Let $x \in \mathcal{N}(P), y \in \mathcal{R}(P)$. For any $\lambda \in \mathbb{C}$,

$$\|\lambda y\|^2 = |\lambda|\|y\|^2$$

$$= \|P(x + \lambda y)\|^2 \quad \text{(since } Px = 0 \text{ and } P\lambda y = \lambda P y = \lambda y)$$

$$\leq \|x + \lambda y\|^2 \quad \text{(since } \|P\| = 1 \implies P \text{ is contractive)}$$

$$= \langle x + \lambda y, x + \lambda y \rangle$$

$$= \|x\|^2 + 2\Re(\overline{\lambda}\langle x, y \rangle) + \|\lambda y\|^2$$

Then $\|\lambda y\|^2 \leq \|x\|^2 + 2\Re(\overline{\lambda}\langle x, y \rangle) + \|\lambda y\|^2$ which gives $\|x\|^2 + 2\Re(\overline{\lambda}\langle x, y \rangle) \geq 0$ for all $\lambda \in \mathbb{C}$. Set $\lambda = t\langle x, y \rangle$ for $t \in \mathbb{R}$ so that $\overline{\lambda} = t\langle y, x \rangle$. Then

$$\|x\|^2 + 2\Re(t\langle x, y \rangle)^2 \geq 0.$$

This inequality holds for all $t \in \mathbb{R}$, hence if we choose $t$ to be a small enough negative number ($t < -\frac{\|x\|^2}{2\|\langle x, y \rangle\|^2}$) then the inequality does not hold unless $\langle x, y \rangle = 0$. Since we began with $x \in \mathcal{N}(P), y \in \mathcal{R}(P)$, we have $\mathcal{N}(P) \perp \mathcal{R}(P)$.

(1. $\implies$ 3.)

$$\langle Px, x \rangle = \langle Px, x - Px + Px \rangle$$

$$= \langle Px, x - Px \rangle + \langle Px, Px \rangle$$

$$= \langle Px, (I - P)x \rangle + \|Px\|^2 \quad = \|Px\|^2 \text{ since } \mathcal{N}(P) = \mathcal{R}(I - P) \perp \mathcal{R}(P)$$

Thus, $\langle Px, x \rangle = \|Px\|^2 \geq 0$

(3. $\implies$ 4.) We saw earlier, in the theorem on sesquilinear and quadratic forms, that an operator $P$ is Hermitian if, and only if $\forall x \in \mathcal{H}, x \mapsto \langle Px, x \rangle \in \mathbb{R}$ which holds since $\langle Px, x \rangle \geq 0$.

(4. $\implies$ 5.) $P$ is Hermitian $P = P^* \implies P$ is normal, $PP^* = P^*P = P^2$.

(5. $\implies$ 1.) If $P$ is normal, then for each $x \in \mathcal{H}$, $\|Px\| = \|P^*x\|$ hence $Px = 0 \iff P^*x = 0$. We thus get that

$$\mathcal{N}(P) = \mathcal{N}(P^*) = \mathcal{R}(P^{**}) = \mathcal{R}(P) = \mathcal{R}(P^\perp).$$

Hence $\mathcal{N}(P) \perp \mathcal{R}(P)$. $\square$

A good way of summarizing the properties of an orthogonal projection is $P = PP^*$ since this implies $P = P^*$ and $P = P^2$. 

2
1.49 Examples (The left shift operator). Let

\[ S : ℓ² \rightarrow ℓ² \quad (Sx)_j = x_{j+1}. \]

It takes the element \( x = (x_1, x_2, \cdots) \in ℓ² \) to \( (x_2, x_3, \cdots) \in ℓ² \). \( ℓ² \) is spanned by the orthonormal basis \( \{δ_s : s \in \mathbb{N}\} \) and \( \langle Sx, x \rangle = \langle x, S^*x \rangle \) for all \( x \in ℓ² \). For the basis vectors, we have for \( s \geq 2 \)

\[
\langle Sδ_s, δ_t \rangle = \langle δ_s, S^*δ_t \rangle \\
\langle δ_{s+1}, δ_t \rangle = \langle δ_s, S^*δ_t \rangle \\
\langle δ_{s+1}, δ_t \rangle = 1 \text{ for } s + 1 = t \text{ and } 0 \text{ otherwise}
\]

Thus \( \langle δ_s, S^*δ_t \rangle = 1 \) for \( s + 1 = t \) and \( 0 \) otherwise,

\[
(S^*δ_{s+1})_s = 1 \implies S^*δ_s = δ_{s-1}.
\]

Note that \( \langle Sδ_1, δ_t \rangle = 0 = \langle δ_s, S^*δ_t \rangle \) for all \( t \) thus \( S^*δ_1 = 0 \). By extending linearly, we see that the adjoint is given by the right shift operator

\[ S^* : ℓ² \rightarrow ℓ² \quad (S^*x)_1 = 0; \quad (S^*x)_j = x_{j-1} \text{ for } j \geq 2 \]

It takes the element \( (x_1, x_2, \cdots) \in ℓ² \) to \( (0, x_1, x_2, \cdots) \in ℓ² \). The map \( S^* \) is an isometry \( (S^*S = \text{id}) \) and is not onto since (thus not unitary) the element \( (x, 0, 0, \cdots) \) has no preimage under \( S^* \). The map \( S^*S \) given by \( (S^*Sx)_1 = 0; \quad (S^*Sx)_j = x_j \) for \( j \geq 2 \) projects orthogonally onto \( R(S^*) \).

2 Spectral Theory

Recall, from linear algebra, the concept of eigenvalues and eigenvectors. These gave us a lot of information about matrices (or operators on finite dimensional vector spaces). In general, for studying operators on Hilbert spaces, a generalized notion called the spectrum is introduced and studied. It is defined as follows. Given an operator \( A \), consider \( T_z = A - z\text{Id}_H \) for \( z \in \mathbb{C} \) and ask if \( T_z \) has a bounded inverse. The resolvent is the set of \( \{z \in \mathbb{C} : T_z \text{ invertible}\} \) and the spectrum is the complement of the resolvent. If \( T_z \) is invertible, the inverse is given by a polynomial in powers of \( A \). Thus the Neumann series \( \sum_i λ_i A^i \) is associated with \( A \), are studied to understand the resolvent and spectrum. However, we will take a slightly different approach in this course to introduce these notions. The main goal here is to understand the behaviour of normal operators, especially unitary and Hermitian ones. Representation theory offers a good framework for generating insight. A representation is a map from some ‘structured space’ to operators on a Hilbert space. We start with a definition of involutive semigroups.

2.1 Definition (Involutive semigroup). A pair \((π : S, \ast)\) of a semigroup \( S \) with an involutive anti-automorphism \( s \mapsto s^\ast \) is called an involutive semigroup.

Note:

- The anti-automorphism gives \((st)^\ast = t^\ast s^\ast\), i.e., it reverses the order of the semigroup operation.

• If 1 is a unit, then 1s = s ∀s ∈ S. In particular, 11* = 1*. Thus, (11*)* = (1*)* gives 11* = 1 and so 1* = 1.

2.2 Definition (Hermitian and unitaries). Elements in \( S_h = \{ s ∈ S : s = s^* \} \) are called hermitian, and elements in \( S_u = \{ s ∈ S : ss^* = s^*s = 1 \} \) are called unitaries. \( S_u \) forms a group, the unitary group of \( S \).

2.3 Examples. 1. If \( S \) is an abelian semigroup and \( s^* = s \), \( s ∈ S \), then \( (S, ∗) \) is an involutive semigroup.

2. If \( G \) is a group, and \( g^* = g^{-1} \), \( g ∈ G \) then \( (G, ∗) \) is an involutive semigroup.

3. \( \mathbb{B}(H) \) with the adjoint operation \( A \mapsto A^* \), \( A ∈ \mathbb{B}(H) \) is an involutive semigroup.

4. The multiplicative semigroup \( \mathbb{C} \setminus \{0\} \) with \( z^* = \bar{z} \), \( z ∈ \mathbb{C} \) is an involutive semigroup.

5. If \( X \) is any set then define \( \mathbb{C}^X := \{ f : X → \mathbb{C} \} \), the set of all maps from \( X \) to \( \mathbb{C} \) with semigroup operation \( (fg)(x) = f(x)g(x) \) and involution \( f^*(x) = \overline{f(x)} \), \( x ∈ X \). \( (\mathbb{C}^X, ∗) \) is an involutive semigroup.

• The Hermitian elements are precisely those functions satisfying \( f(x) = \overline{f(x)} \) ∀\( x ∈ X \), i.e., the real-valued functions \( (\mathbb{C}^X_h = \mathbb{R}^X) \).

• The identity 1 satisfies \( f(x)1(x) = f(x) \) ∀\( x ∈ X \). \( 1(x) = 1 \) ∀\( x ∈ X \).

• The unitaries are \( \mathbb{C}^X_u = \{ f : X → \mathbb{C} : |f(x)| = 1 \) ∀\( x ∈ X \} \). orthogonal projections given by \( f(x) = |f(x)|^2 \) ∀\( x ∈ X \) are precisely the functions with values 0 and 1.