Last Time

- Characterization of orthogonal projection.
- Isometries v/s orthogonal projections.
- Semigroups

Warm up: Let $X$ be a set, consider $C^X \equiv \{ f : X \to \mathbb{C} \}$ such that $(fg)(x) = f(x)g(x)$, $f^*(x) = \overline{f(x)}$.

If $f$ is an orthogonal projection, then $f^*f = f \iff f\overline{f} = f \iff |f|^2 = f \iff f$ has values 0 or 1.

4.5 Question. What are unitaries in $C^X$?

The identity of $C^X$ is $e(x) = 1$. Unitary group of the semigroup is $C^X_u = \{ f : X \to \mathbb{C} : \forall x \in X, |f(x)| = 1 \}$

4.6 Definition. A representation of an involutive semigroup $S$ is a homomorphism $\pi : S \to B(\mathcal{H})$ where $\mathcal{H}$ is a complex Hilbert space, and for which $\pi(s^*) = (\pi(s))^*$ for each $s \in S$. We also write $(\pi, \mathcal{H})$ if we need to keep track of $\mathcal{H}$.

If $G$ is a group with $g^* = g^{-1}$ and $\pi(1) = \text{id}_\mathcal{H}$ then the representation is called unitary.

4.7 Remark. 1. From $g^*g = gg^* = 1$, we get $\pi(g) = B(\mathcal{H})_u$, so a unitary (group) representation maps into the unitary group of Hilbert space (since, $\pi(s^*s) = \pi(s^{-1}s) = \pi(1) = \text{id}_\mathcal{H}$ and $\pi(s^*)\pi(s) = \pi(s^{-1}\pi(s) = (\pi(s))^{-1}\pi(s) = \text{id}_\mathcal{H}$).

2. In general, if $1$ is a unit of $S$ and $\pi$ is a representation, then we only know that $\pi(1) = (\pi(1)(\pi(1))^* \pi(1)$ is an orthogonal projection. So $\pi(1)$ is an orthogonal projection based on the characterization of orthogonal projection in class.

4.8 Definition. Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of an involutive semigroup are called equivalent if there is a unitary $u \in B(\mathcal{H}, \mathcal{H}')$ and for each $s \in S, u \circ \pi(s) = \pi'(s) \circ u$. In this context, $u$ is called interwining operator. (Since, $u^*u \circ \pi(s) = u^* \pi'(s) u \implies \pi(s) = u^* \pi'(s) \circ u$. Conversely, $u \circ \pi(s) \circ u^* = \pi'(s) \circ u \circ u^* \implies \pi'(s) = u \circ \pi(s) \circ u^*$). This shows that we can learn all about $\pi$ from $\pi'$ and vice versa.
4.9 Remarks. 1. Let \( S = Z \) (additive) and \( s^* = s \), and let \((\pi, \mathcal{H})\) be the unitary representation of \( S \), so \( \pi(1) = u \in (B(\mathcal{H}))_u \) and then \( \pi(0) = u^0 \equiv \text{id}_\mathcal{H}, u^{-n} \equiv (u^{-1})^n \). Conversely, given the unitary operator \( u \) on \( \mathcal{H}, \pi(n) = u^n \) defines a representation of \( Z \) on \( \mathcal{H} \).

Studying \( U \) and studying the associated representation concerns the same information.

2. Let \( S = (\mathbb{N}_0, +) \) such that \( S = S^* \). If \((\pi, \mathcal{H})\) is a representation of \( S \), then \( A = \pi(1) \), then \( \pi(n) = A^n \) and \( A = A^* \). Studying representations of \( S \) is equivalent to studying each \( A \in (B(\mathcal{H}))_n \).

3. Let \((\mathbb{N}_0 \times \mathbb{N}_0, +)\) with \((m, n)^* = (n, m)\). Then this is an abelian involutive semigroup. If \((\pi, \mathcal{H})\) is a representation, \( A = \pi(1, 0) \) defines \( \pi \). Assuming \( \pi(0, 0) = \text{id}_\mathcal{H} \), we have, \( \pi(0, 1) = \pi((1, 0))^* = A^* \), and \( AA^* = \pi(1, 0)\pi(0, 1) = \pi(1, 1) = \pi(0, 1)\pi(1, 0) = A^*A \).

So, \( A \) is normal, and for each \( n, m \in \mathbb{N}_0 \), \( \pi(n, m) = A^n(A^*)^m \)

We have thus converted between studying unitary, hermitian and normal operators and study of representations of semigroups.

4.10 Question. Give a representation, can it be reduced to fundamental ingredients/parts?

4.11 Definition. 1. A representation of an involutive semigroup is called non-degenerate if \( \pi(S)\mathcal{H} = \{\pi(s)v : s \in S, v \in \mathcal{H}\} \) is total or equivalently \( \pi(S)\mathcal{H} \) is dense in \( \mathcal{H} \). This is the case if \( S \) has a unit 1 and \( \pi(1) = \text{id}_\mathcal{H} \) (Since \( \pi(1) \) is an orthogonal projection, in general it projects onto a closed subspace of \( \mathcal{H} \). Therefore, \( \pi(S)\mathcal{H} \) is the subset of the closed subspace which may or may not be in \( \mathcal{H} \)).

2. A representation \((\pi, \mathcal{H})\) of an involutive semigroup is called cyclic, if there is \( v \in \mathcal{H} \) such that \( \pi(S)v = \{\pi(s)v : s \in S\} \) is total.

3. A representation \((\pi, \mathcal{H})\) of an involutive semigroup is called irreducible if 0 and \( \mathcal{H} \) are the only closed subspaces that are invariant under \( \pi(S) \).

4.12 Lemma. Let \((\pi, \mathcal{H})\) be a representation of an involutive semigroup \( S \), \( E \subset \mathcal{H} \) be a subspace, and \( P_E \) be the orthogonal projection onto \( E \), then the following are equivalent:

1. \( E \) is invariant under \( S \).
2. \( E^\perp \) is invariant under \( S \).
3. \( P_E\pi(S) = \pi(S)P_E \)

Proof. (1) \( \implies \) (2)

By \((\pi(s))^* = \pi(s^*)\) We know \( E \) is invariant under \( \pi(s) \) if and only if \( E^\perp \) is invariant under \( \pi(s^*) \). Hence, \( E \) is invariant under \( \pi(S) \) if and only if \( E^\perp \) is invariant under \( \pi(S) \).

(2) \( \implies \) (3)

Let \( v \in \mathcal{H}, v_E = P_Ev \). For \( s \in S \)
\begin{align*}
P_E\pi(s)\nu &= P_E\pi(s)\left(\nu_E + \nu - \nu_E\right) \\
&= P_E(\pi(s)\nu_E + \pi(s)(\nu - c_E)) \\
&= P_E\pi(s)\nu_E \\
&= \pi(s)P_E\nu_E \\
&= \pi(s)P_E(\nu_E + \nu - \nu_E)
\end{align*}

We conclude $P_E\pi(s) = \pi(s)P_E$ for each $s \in S$. \hfill \qed