Last time

- Characterization of orthogonal projections.
- Isometries versus orthogonal projections.
- Semigroups.

Warm up:

Let $X$ be a set. Last class, we considered the space $\mathbb{C}^X \equiv \{ f : X \to \mathbb{C} \}$, as a semigroup, where the operation is pointwise multiplication: $(fg)(x) = f(x)g(x)$, and the anti-automorphism is given by the complex conjugate: $f^*(x) = \overline{f(x)}$. We might consider the following:

1.6 Question. What are the unitaries in $\mathbb{C}^X$? What about the orthogonal projections?

Well, the identity in $\mathbb{C}^X$ is $e(x) \equiv 1$. Hence the unitary group of the semigroup is

$$(\mathbb{C}^X)_u = \{ f : X \to \mathbb{C} : \text{for all } x \in X, |f(x)| = 1 \}$$

since $f(x)f^*(x) = f(x)\overline{f(x)} = |f(x)|^2$. As for the orthogonal projections, they are the 0 or 1 valued functions as they satisfy $f(x)\overline{f(x)} = f(x)$ from our characterization of orthogonal projections.

A bit of wisdom before we continue:

*If anyone asks you in anything in math, you just say zero. And if they say no, then you say one.*

- Professor Bodmann

1.7 Definition. A representation of an involutive semigroup $S$ is a homomorphism $\pi : S \to B(\mathcal{H})$, that is,

$$\pi(ab) = \pi(a)\pi(b)$$

for all $a, b \in S$, where $\mathcal{H}$ is the complex Hilbert space, and for which

$$\pi(s^*) = (\pi(s))^* \quad \text{for each } s \in S.$$
1.8 Question. Why is it a ‘good’ idea to call such representations unitaries?

From \( g^*g = gg^* = 1 \), we have \( \pi(g) \in (B(H))_u \), so a unitary (group) representation maps into the unitary group of a Hilbert space.

- In general, if \( 1 \) is a unit of \( S \) and \( \pi \) are representation, then we only know \( \pi(1) = \pi(1)(\pi(1))^* \), which implies that \( \pi(1) \) is an orthogonal projection.

1.9 Definition. Two representations \( (\pi, H) \) and \( (\pi', H') \) of an involutive semigroup \( S \) are called equivalent if there exists a unitary \( u \in B(H, H') \) and for each \( s \in S \),

\[
u \circ \pi(s) = \pi'(s) \circ u.
\]

- In this context, \( u \) is called an intertwining operator.

1.10 Exercise. Show that the above definition induces an equivalence relation \( \sim \) on representations of involutive semigroups.

Proof sketch. Reflexive. \( \pi \sim \pi \) since for each unitary \( u \in B(H) \) and \( s \in S \), we have

\[
u \circ \pi(s) \circ u^{-1} = \pi(s) \Rightarrow u \circ \pi(s) = \pi(s) \circ u.
\]

Symmetric. Suppose \( (\pi, H) \) and \( (\pi', H') \) are two representations of an involutive semigroup \( S \) such that \( \pi \sim \pi' \). Then there exists a unitary \( u \in B(H, H') \) such that

\[
u \circ \pi(s) = \pi'(s) \circ u \quad \iff \quad \pi(s) = u^{-1} \circ \pi'(s) \circ u \\
\quad \iff \quad \pi(s) \circ u^{-1} = u^{-1} \circ \pi'(s),
\]

hence, \( \pi' \sim \pi \).

Transitive. We leave it to the reader.

1.11 Remark. (a) Let \( S = \mathbb{Z} \) be the group of integers under addition, with \( s^* = -s \), and let \( (\pi, H) \) be a unitary representation of \( S \). Then \( \pi(1) \) is a unitary \( \pi(1) = u \in (B(H))_u \), hence

\[
\pi(n) = \pi(1)^n = u^n \quad \text{for each } n \in \mathbb{Z}
\]

with \( \pi(0) = u^0 = id_H \). Also,

\[
u^{-n} \equiv (u^{-1})^n = (u^*)^n \quad \text{for each } n \in \mathbb{Z}.
\]

Conversely, given a unitary \( u \) on \( H \), \( \pi(1) = u^n \) defines a representation of \( \mathbb{Z} \) on \( H \). We see that studying \( u \) and studying the associated representations concern the same information.

(b) Let \( S = (\mathbb{N}_0, +) \), \( s^* = s \). If \( (\pi, H) \) is a representation of \( S \) and \( A = \pi(1) \), then for \( n \in \mathbb{N} \) we have

\[
\pi(n) = A^n \quad \text{and} \quad A = A^*.
\]

We observe that studying representations of \( S \) is equivalent to studying each \( A \in (B(H))_h \).
(c) Let \( S = (\mathbb{N}_0 \times \mathbb{N}_0, +) \) with \((n, m)^* = (m, n)\). If \((\pi, \mathcal{H})\) is a representation of \( S \), then we claim that \( A = \pi(1, 0) \) determines \( \pi \), assuming \( \pi(0, 0) = \text{id}_\mathcal{H} \). We have
\[
\pi(0, 1) = \pi((1, 0)^*) = (\pi(1, 0))^* = A^*,
\]
and hence
\[
AA^* = \pi(1, 0)\pi(0, 1) = \pi(1, 1) = \pi(0, 1)\pi(1, 0) = A^*A,
\]
which implies that \( A \) is normal, and for all \( m, n \in \mathbb{N}_0 \), \( \pi(m, n) = A^m(A^*)^n \). This shows the claim. Conversely, we see each normal operator defines a representation of this group.

We have thus converted between studying unitary, Hermitian, and normal operators and the study of representations of specific semigroups.

1.12 Question. Given a representation, can it be reduced to the fundamental ingredients/parts?

1.13 Definition. (a) A representation \((\pi, \mathcal{H})\) of an involutive semigroup is called non-degenerate if
\[
\pi(S)\mathcal{H} = \{\pi(s)v : s \in S, v \in \mathcal{H}\}
\]
is total, or equivalently, \( \pi(S)\mathcal{H} \) is dense in \( \mathcal{H} \).
(This is the case if \( S \) has a unit \( 1 \) and \( \pi(1) = \text{id}_\mathcal{H} \).)

(b) A representation \((\pi, \mathcal{H})\) of an involutive semigroup is called cyclic if there is \( v \in \mathcal{H} \) such that
\[
\pi(S)v = \{\pi(s)v : s \in S\}
\]
is total.

(c) A representation \((\pi, \mathcal{H})\) of an involutive semigroup is called irreducible if \( \{0\} \) and \( \mathcal{H} \) are the only closed subspaces that are invariant under \( \pi(S) \).

1.14 Remark. Irreducible representations are always cyclic. We will show this later.

1.15 Lemma. Let \((\pi, \mathcal{H})\) be a representation of an involutive semigroup \( S \), \( E \subset \mathcal{H} \) a closed subspace, and \( P_E \) an orthogonal projection onto \( E \). The following are equivalent:

1. \( E \) is invariant under \( \pi \).
2. \( E^\perp \) is invariant under \( \pi \).
3. \( P_E\pi(S) = \pi(S)P_E \).

Proof strategy: Show (1) and (2) are equivalent, then use them to show that (3) holds, and finally show that (3) implies (1).
Proof. (1) $\iff$ (2). We start by observing that $\pi(s)^* = \pi(s^*)$ implies $E$ is invariant under $\pi(s)$ if and only if $E^\perp$ is invariant under $\pi(s^*)$. Hence, $E$ is invariant under $\pi(s)$ if and only if $E^\perp$ is invariant under $\pi(s)$. Now assuming that $E$ is invariant under $\pi$, $E$ is invariant under $\pi(s)$ for all $s \in S$, and equivalently, this holds true for all $s^* \in S$. This is equivalent to $E^\perp$ being invariant under $(\pi(s^*))^* = \pi(s)^{**} = \pi(s)$ for all $s^* \in S$, which is equivalent to $E^\perp$ invariant under $\pi(s)$ for all $s \in S$. This proves that $E$ is invariant under $\pi$ if and only if $E^\perp$ is invariant under $\pi$.

(1) and (2) $\Rightarrow$ (3). Let $v \in H$ and set $v_E = P_EV$. For $s \in S$,
\[
P_E \pi(s) v = P_E \pi(s) (\underbrace{v_E}_{E} + \underbrace{v - v_E}_{E^\perp})
= P_E (\pi(s) v_E + \pi(s) (v - v_E))
= P_E \pi(s) v_E
= \pi(s) P_E v.
\]
We conclude $P_E \pi(s) = \pi(s) P_E$ for all $s \in S$. We delay the rest of the proof until the next lecture. \qed