Non-degenerate representations, finite dimensional representations, and representations with intertwining operators

Lecture Notes from September 27, 2022
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Last Time
- Restricting a representation to make it non-degenerate
- Direct sum representations
- How to get cyclic representations

Warm up
Recall the Lemma on how to get cyclic representations from last time:

0.0 Lemma. Take \( v \in \mathcal{H} \) and restrict \( \pi \) to \( \mathcal{H}_v := \text{span} \, \pi(S)v \), then \( \pi \) is cyclic when restricted to \( \mathcal{H}_v \), and if \( \pi \) is non-degenerate, we have \( v \in \mathcal{H}_v \).

We now compare the above Lemma with the semigroup representation

\[
\pi : \mathbb{N} \to B(l^2), \\
\pi(1) = S^*,
\]

where \( S^* \) is the right shift function. We see that, with this example, we ignore the involution condition \( \pi(s^*) = (\pi(s))^*, s \in S \), since

\[
(\pi(1))^* = (S^*)^* \neq S^* := \pi(1) = \pi(1^*).
\]

Thus, when we choose \( v = e_1 \), we have \( \pi(n)e_1 = e_{1+n} \), so

\[
\pi(N)e_1 = \{e_{1+n} : n \in \mathbb{N}\} = \{e_m : m \geq 2\},
\]

which implies \( \mathcal{H}_{e_1} := \text{span} \, \pi(N)e_1 \) is the orthogonal complement of \( e_1 \), \( \{e_1\}^\perp \). Since \( \mathcal{H}_{e_1} \) contains elements starting with at least one zero, we see that \( e_1 \notin \mathcal{H}_{e_1} \), and \( \pi \) restricted to \( \mathcal{H}_{e_1} \) has an invariant subspace \( \mathcal{H}' : \{x \in l^2 : x_1 = x_2 = 0\}, \mathcal{H}' \neq \{e_1\}^\perp \) (specifically, \( \mathcal{H}' \subset \{e_1\}^\perp \)), and \( \mathcal{H}' \neq \{0\} \), meaning \( \pi \) restricted to \( \mathcal{H}_{e_1} \) is not irreducible.
1 Non-degenerate Representations

We begin by restating Zorn’s Lemma, to be used as the main proof device for the theorem in this section:

1.1 Lemma. Suppose a partially ordered set $P$ has the property that every chain in $P$ has an upper bound in $P$. Then the set $P$ contains a maximal element.

1.2 Theorem. The representation $\pi$ of an involutive semigroup $S$ is non-degenerate if and only if $\pi$ is the direct sum of cyclic representations.

Proof. Let $\pi$ be non-degenerate. Since $\mathcal{H} = \overline{\text{span} \pi(S)\mathcal{H}}$, there is a $0 \neq v \in \mathcal{H}$ for which $H_v := \overline{\text{span} \pi(S)v} \neq 0$.

Let $M$ be the set of all families $\{H_j\}_{j \in J}$ such that $H_j \perp H_k$ if $j \neq k$, and each $H_j$ is a closed subspace. We observe that, because there is a non-zero $v \in \mathcal{H}$ so that $H_v$ is nontrivial, by the assumption that $\pi$ is non-degenerate, $M$ is not zero, and there is a partial ordering on $M$: Let $K := (K_m), m \in \mathbb{N}$ be a chain in $M, K_m := \{H_j^m\}, j \in J^m$. We say

$$K_n \leq K_{n+1}$$

if for any $H_v^m \in K_m, H_v^m \in K_{n+1}$. Then the upper bound for $K$ is

$$U := \bigcup_m \{H_v^m : H_v^m \in K_m\}.$$

To see $U \in M$, take any two elements $H_v^n, H_i^m \in U$. Then there must be $K_n$ and $K_m$ so that $H_v^n \in K_n$ and $H_i^m \in K_m$. Without loss of generality, assume $K_m \leq K_n$, then $H_i^m \in K^n$ and thus $H_i^m \perp H_v^n$, so $U$ is an element in $M$.

Now, since every chain $K$ has an upper bound in $M$, Zorn’s lemma gives us a maximal element $\{H_j\}_{j \in J_{\text{max}}}$. Letting $H_1 = \sum_{j \in J_{\text{max}}} H_j$, then $H_1^\perp$ is an invariant subspace (by Lemma from September 22). If $H_1 \neq \mathcal{H}$, then since $\pi$ is non-degenerate, there is $0 \neq v \in H_1^\perp$ such that $H' := \overline{\text{span} \pi(S)v} \neq 0$, and $\{H'\}$ and $\{H_j\}_{j \in J_{\text{max}}}$ form an orthogonal family of closed subspaces, contradicting the maximality assumption. Therefore, the subspace associated with the maximal element in $M, H_1$, exhausts the whole space $\mathcal{H}$, i.e. $H_1 = \mathcal{H}$, and $H_1^\perp = \{0\}$.

Conversely, if $(\pi, \mathcal{H})$ is a direct sum of cyclic representations $(\pi_j, H_j)$, then $\sum_{j \in J} H_j$ is dense in $\mathcal{H}$. Since each representation $\pi_j$ is cyclic,

$$H_j \subset \overline{\text{span} \pi(S)H_j} \subset \overline{\text{span} \pi(S)\mathcal{H}}.$$

Thus,

$$\sum_{j \in J} H_j \subset \overline{\text{span} \pi(S)\mathcal{H}}.$$

Since $\overline{\text{span} \pi(S)\mathcal{H}}$ is closed and $\sum_{j \in J} H_j$ is dense in $\mathcal{H}$, $\overline{\text{span} \pi(S)\mathcal{H}} = \mathcal{H}$, so $S$ is non-degenerate.
2 Finite Dimensional Representations

2.1 Theorem. Each finite dimensional representation $\pi$ of an involutive semigroup $S$ is a direct sum of irreducible representations.

Proof. If $(\pi, \mathcal{H})$ is irreducible, we have nothing to show.
Otherwise, there is a reducible subspace $\mathcal{H}_1, \mathcal{H}_1 \neq 0, \mathcal{H}_1 \neq \mathcal{H}$ which is invariant. We obtain $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ where both subspaces are invariant and have dimensions less than dimension of $\mathcal{H}$.

Next, we argue with induction over the dimension of $\mathcal{H}$:
For $\dim \mathcal{H} = 1$, $\pi$ is irreducible since either $\mathcal{H}_1$ or $\mathcal{H}_1^\perp$ has to be $\{0\}$ or $\mathcal{H}$.
If $\dim \mathcal{H} > 1$, splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ and applying induction hypothesis to subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ gives that, each of these subspaces is a direct sum of subspaces on which $\pi$ acts irreducibly. One can also envision a tree of splittings. After each split, we check if each summand is irreducible. If not, we keep splitting and checking irreducibility again. After an finite amount of splittings (since $\mathcal{H}$ is of finite dimension), we must arrive at the irreducible representations. $\square$

3 Representations with intertwining operator

We first recall the definition of intertwining operators:

3.1 Definition. An operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is called intertwining if, for two representations $(\pi, \mathcal{H}), (\pi', \mathcal{H}')$ of an involutive semigroup $S$,

$$U \circ \pi(s) = \pi'(s) \circ U$$

for all $s \in S$.

3.2 Lemma. Given a representation $(\pi, \mathcal{H})$ of a semigroup with involution $S$, $A \in \mathcal{B}(\mathcal{H})$ an intertwining operator, and

$$\mathcal{H}_\lambda(A) := \{v \in \mathcal{H} : Av = \lambda v\},$$

then $\mathcal{H}_\lambda(A)$ is invariant under $S$.

Proof. Using the above definition, for $v \in \mathcal{H}_\lambda(A), s \in S$

$$A(\pi(s)v) = \pi(s)Av = \pi(s)\lambda v = \lambda(\pi(s)v),$$

which implies that $\pi(s)v \in \mathcal{H}_\lambda(A)$, so $\mathcal{H}_\lambda(A)$ is invariant under $S$. $\square$

Next, we consider the case where $S$ is abelian.

3.3 Theorem. If $S$ is abelian, then each irreducible finite dimensional representation is one dimensional.
Proof. Consider $s \in S$ and $\pi(s)$. By $\mathcal{H}$ being complex, the characteristic polynomial has at least one root, so there is an $\lambda \in \mathbb{C}$ such that $\mathcal{H}_\lambda(\pi(s)) \neq 0$.

Since $S$ is abelian, $\pi(s)$ intertwines $\pi$, and by the above Lemma, $\mathcal{H}_\lambda(\pi(s))$ is invariant under $S$. By the irreducibility of $\pi$, $\mathcal{H}_\lambda(\pi(s)) := \{v \in \mathcal{H} : \pi(s)v = \lambda v\} = \mathcal{H}$, so

$$\pi(s) = \lambda \text{id}_\mathcal{H}.$$ 

We conclude that $\pi(S) \subset \mathbb{C} \text{id}_\mathcal{H}$, but $\pi$ is irreducible, i.e. $\{0\}$ and $\mathcal{H}_\lambda(\lambda \text{id}_\mathcal{H})$ are the only closed subspaces that are invariant under $\pi(S)$, so there cannot be an orthogonal projection $P$ so that $0 \neq P \neq \text{id}_\mathcal{H}$ with $P\pi(s) = \pi(s)P$ (by the Lemma on September 20), and hence $\dim \mathcal{H}$ must be 1, since if such $P$ existed, then $\mathcal{H}$ would be the direct sum of two nontrivial orthogonal subspaces, making $\dim \mathcal{H} \geq 2$. \[\square\]