Last time

- Restricting a representation to make it non-degenerate
- Direct sum representations
- How to get cyclic representations

Recall the following:

1.6 Lemma. Take $v \in \mathcal{H}$, restrict $\pi$ to $\mathcal{H}_v \cong \text{span} \pi(S)v$. The $\pi$ is cyclic when restricted to $\mathcal{H}_v$. If $\pi$ is non-degenerate then $v \in \mathcal{H}_v$.

Warm up:

1.7 Question. What happens if $S$ is not involutive?

Compare the lemma with the following example: $\pi : \mathbb{N} \to B(l^2)$. Where $\pi(1) = S^*$ is the right shift.

Let $v = e_1$. Then, $\pi(n)v = e_{1+n}$. Thus, $\text{span} \pi(S)v = \{0, x_2, x_3, \ldots\} = \{e_1\}^\perp = \mathcal{H}_{e_1}$, $e_1 \notin \{e_1\}^\perp$. Also, $\pi|_{\mathcal{H}_{e_1}}$ has infinitely many invariant subspaces. One of them is $\mathcal{H}^2 = \{x \in l^2 : x_1 = x_2 = 0\}$. $\mathcal{H}^2 \neq \{e_1\}^\perp$, and $\mathcal{H}^2 \neq \{0\}$. Thus, $\pi|_{\mathcal{H}_{e_1}}$ is not irreducible. Also, $\pi|_{\mathcal{H}_{e_1}}$ can be seen to not be cyclic. If we take $w \in \mathcal{H}_{e_1}$ then for all $s \in S$, $\pi(s)w \subset \mathcal{H}^2$. In particular, $e_2 \notin \text{span} \pi(S)w$.

We now turn back to the case of representations of involutive semigroups and the linear structure of nondegenerate representations.

1.8 Theorem. The representation of an involutive semigroup, $S$, is nondegenerate iff it is the direct sum of cyclic representations.

Proof. Let the representation be non-degenerate. If it is not, we can "make" it non-degenerate by the first theorem of September 22. That is, we can remove, in the sense of direct sums, the intersection of the kernels of $\pi(s)$ for each $s \in S$.

Since our representation is nondegenerate, $\mathcal{H} = \text{span} \pi(S)\mathcal{H}$, so there exists a vector $v \in \mathcal{H}$ for which $\text{span} \pi(S)v$ is not zero. We are looking to apply Zorn’s Lemma as a countable process need not “exhaust” the Hilbert Space $\mathcal{H}$. Let $\mathcal{M}$ be the set of all indexed families of closed mutually orthogonal cyclic subspaces of $\mathcal{H}$ that are invariant under $\pi$. That is, $\mathcal{M}$ is the set of $\{\mathcal{H}_j\}_{j \in J}$ such that $\mathcal{H}_j \perp \mathcal{H}_k$ for $k \neq j$, each $\mathcal{H}_j$ is a closed subspace, is cyclic, and is invariant
under \( \pi \). We can define a partial order on \( \mathcal{M} \) by \( A_1 < A_2 \) if \( A_1 \subset A_2 \). Here we are saying that if \( \mathcal{H}_j \in A_1 \) then \( \mathcal{H}_j \in A_2 \). We note that for each chain, \( \mathcal{K} = (K_m) \), \( K = \bigcup_{m} \{ \mathcal{H}_j : \mathcal{H}_j \in K_m \} \) is an upper bound. This follows from its construction. If \( \mathcal{H}_j \in K_m \) for any \( m \in \mathbb{N} \), then \( \mathcal{H}_j \in K \). Hence, \( K_m < K \) for all \( m \in \mathbb{N} \). If \( \mathcal{H}_j \) and \( \mathcal{H}_k \) are in \( K \) then there is an \( m \in \mathbb{N} \) that contain them both. Thus, if \( \mathcal{H}_j \neq \mathcal{H}_k \), \( \mathcal{H}_j \perp \mathcal{H}_k \). Also, for the same reason, these \( \mathcal{H}_j \) are cyclic and invariant under \( \pi \), which shows \( K \) is indeed an element of \( \mathcal{M} \). Thus, by Zorn’s Lemma, there exists a maximal element \( A \in \mathcal{M} \). By maximal we mean, if \( A < B \) for some \( B \in \mathcal{M} \) then \( B = A \). Denote \( A = \{ \mathcal{H}_j \}_{j \in J_{\max}} \). Let \( \mathcal{H}_1 = \sum_{j \in J_{\max}} \mathcal{H}_j \). Each \( \mathcal{H}_j \) is an invariant subspace, so \( \mathcal{H}_1 \) is invariant. This took me ten minutes of thinking before I concluded it was obvious. If you are like me then let me spare you some of those minutes. If \( w \in \mathcal{H}_1 \) then there exists a sequence \( w_i \) in \( \sum_{j \in J_{\max}} \mathcal{H}_j \) converging to \( w \). Thus, for any \( s \in S \), \( \pi(s)w_i \in \sum_{j \in J_{\max}} \mathcal{H}_j \) for all \( i \) by the invariance of each \( \mathcal{H}_j \) and the linearity of \( \pi(s) \). By the continuity of \( \pi(s) \), \( \pi(s)w_i \) converges to \( \pi(s)w \). Hence, \( \pi(s)w \in \sum_{j \in J_{\max}} \mathcal{H}_j \). Then, \( \mathcal{H}_1^\perp \) is an invariant subspace. If \( \mathcal{H}_1 \neq \mathcal{H} \), then there exists a nonzero \( v \in \mathcal{H}_1^\perp \). Let \( \mathcal{H}_c = \text{span} \{ \pi(S)w \} \). This closed subspace is invariant and cyclic by construction. Thus, \( \{ \mathcal{H}_j \} \cup \{ \mathcal{H}_j \}_{j \in J_{\max}} \) is an orthogonal family of closed, invariant, cyclic subspaces that properly contains \( A \). This contradicts Zorn’s lemma. Therefore, \( \mathcal{H}_1 = \mathcal{H} \).

Conversely, if \((\pi, \mathcal{H})\) is a direct sum of cyclic representations, \((\pi_j, \mathcal{H}_j)\), then \( \sum_{j \in J} \mathcal{H}_j \) is dense in \( \mathcal{H} \). The representation of each \( \pi_j \) is cyclic, so

\[
\mathcal{H}_j \subset \text{span} \{ \pi(S) \mathcal{H}_j \} \subset \text{span} \{ \pi(S) \mathcal{H} \}.
\]

Hence, if we sum over \( j \) and take the closure, we obtain

\[
\mathcal{H} = \sum_{j \in J} \mathcal{H}_j \subset \sum_{j \in J} \text{span} \{ \pi(S) \mathcal{H}_j \} \subset \sum_{j \in J} \text{span} \{ \pi(S) \mathcal{H} \} = \text{span} \{ \pi(S) \mathcal{H} \} \subset \mathcal{H}.
\]

Relying on the the maxim \( \mathcal{H} = \mathcal{H} \), we have shown \( \mathcal{H} = \text{span} \{ \pi(S) \mathcal{H} \} \), so \( \mathcal{H} \) is nondegenerate. \( \Box \)

Having done the 'hard' work of unrestricted dimensions, we now turn to the finite dimensional case. In this setting we will arrive at direct sums of irreducible representations, which is a stronger condition than cyclic.

1.9 Remark. A nontrivial irreducible representation of an involutive semigroup is cyclic, but a cyclic representation need not be irreducible.

Proof. Suppose we have a nontrivial irreducible representation \((\pi, \mathcal{H})\) of an involutive semigroup \( S \). That is, the only invariant subspaces of \( \pi \) are \( \{0\} \) and \( \mathcal{H} \) and there exists an \( s \in S \) and a \( v \in \mathcal{H} \) such that \( \pi(s)v \neq 0 \). Then, \( W = \text{span} \{ \pi(s)v \} = \mathcal{H} \). This follows from the fact that \( W \) is a nonzero closed subspace of \( \mathcal{H} \) that is invariant under \( \pi \). However, given \( A = [e_3, e_1, e_2] \), the set \( \{1, A, A^2\} \) is a cyclic representation of \( \mathbb{Z}_3 \) in \( \mathcal{H} = \mathbb{R}^3 \). It is cyclic under \( e_1 \) because for any \( w \in \mathcal{H} \), \( w = w_1e_1 + w_2A^2e_1 + w_3Ae_1 \). It is not irreducible because the line in \( \mathbb{R}^3 \) defined by \( \{av : a \in \mathbb{R}, v = (1,1,1)^t\} \) is invariant under \( \pi \) because \( Av = v \). \( \Box \)
1.10 Theorem. Each finite dimensional representation of an involutive semigroup, $S$, is a direct sum of irreducible representations.

Proof. If $(\pi, \mathcal{H})$ is irreducible, there is nothing to prove.

Suppose $(\pi, \mathcal{H})$ is not irreducible. Then, there exists an invariant subspace $\mathcal{H}_1$, such that $\mathcal{H}_1 \neq \{0\}$ and $\mathcal{H}_1 \neq \mathcal{H}$. We also have that $\mathcal{H}_1^\perp$ is invariant by the Lemma from September 20, and $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_1^\perp$.

We are now ready to argue with induction over $\dim \mathcal{H}$. If $\dim \mathcal{H} = 1$ then $\pi$ is irreducible. If $\dim \mathcal{H} > 1$ we can split the Hilbert space $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_1^\perp$ into two subspaces each with dimension at least 1 and at most $\dim \mathcal{H} - 1$. We now ask if $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ are irreducible. If either of them are not, we can again split, which will again reduce the dimension by at least 1. Since $\dim \mathcal{H}$ is finite, this process will eventually terminate with a direct sum of prime irreducible subspaces. Restricting $\pi$ to each of these irreducible subspaces forms the direct sum of irreducible representations. □

We continue towards a description of the structure of the irreducible representations. First, a lemma.

1.11 Lemma. Given a representation of an involutive semigroup, $S$, and an intertwining operator $A \in B(\mathcal{H})$ an intertwining operator, the closed subspace $\mathcal{H}_\lambda(A) = \{v \in \mathcal{H} : Av = \lambda v\}$ is invariant under $S$.

Proof. For $v \in \mathcal{H}_\lambda(A)$, and $s \in S$,

$$A\pi(s)v = \pi(s)Av = \lambda \pi(s)v.$$  

Hence, $\pi(s)v \in \mathcal{H}_\lambda(A)$, so $\mathcal{H}_\lambda(A)$ is invariant under $S$. □

We are now prepared to describe the irreducible representations of abelian involutive semigroups.

1.12 Theorem. If $S$ is abelian, then each irreducible finite representation is 1-dimensional.

Proof. Let $s \in S$. Then, the operator $\pi(s)$ has a characteristic polynomial, which, since $\mathcal{H}$ is complex, has at least one root. Thus, there exists a $\lambda \in \mathbb{C}$ such that $\mathcal{H}_\lambda(\pi(s)) \neq \{0\}$. Since $S$ is abelian, $\pi(s)$ intertwines and hence $\mathcal{H}_\lambda(\pi(s))$ is invariant under $\pi$. By the irreducibility of $\pi$, $\mathcal{H}_\lambda(\pi(s)) = \mathcal{H}$ so $\pi(s) = \lambda \text{Id}_\mathcal{H}$. We conclude $\pi(S) \subset \text{Cld}_\mathcal{H}$.

If the dimension of $\mathcal{H}$ is greater than one, then we can take a $v \neq 0 \in \mathcal{H}$ and form a one dimensional subspace of $\mathcal{H}$ by $\text{span}(v) = V$. However, for any $s \in S$, $\pi(s)v = \lambda v$, so $V$ is invariant under $\pi$. But $V$ is one dimensional and $\mathcal{H}$ is not, which contradicts the irreducibility of $\pi$. Therefore, the dimension of $\mathcal{H}$ is one. □