Lecture Notes from October 10, 2022

taken by Nick Fularczyk

Last time

- characterization of representations of abelian semigroups

Warm up:

1.6 Question. Given $G = (\mathbb{Z}, +)$ and $s^* = -s$, then show

$$\hat{G} \cong S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

To see this, assume $\chi : G \mapsto \mathbb{C}$ is a character. Then, we know that

$$\chi(0) = |\chi(0)|^* \chi(0),$$

so $\chi(0) \in \{0, 1\}$. Moreover, $\chi(n) = \chi(n + 0) = \chi(0) \chi(n)$ for each $n \in \mathbb{Z}$, and $\chi$ is not identically zero by definition of a character. Hence, $\chi(0) = 1$. Furthermore,

$$\chi(n) = \begin{cases} [\chi(1)]^n & n \in \mathbb{N}_0 \\ [\chi(1)^*]^n & n < 0 \end{cases}$$

and $(\chi(1))^* \chi(1) = \chi(0)$. Therefore, if we let $z = \chi(1)$, we see that $zz = 1$, so $|z| = 1$. Hence, every $\chi$ is characterized by $\chi(1) = z \in S^1$. Conversely, given any $z \in S^1$ assigning $\chi(1) = z$ yields a character on $G$.

1.7 Question. Given $S = (\mathbb{N}_0, +)$ and $s^* = s$, then show that $\hat{S}_0 \cong \mathbb{R}$

Note, if $\chi \in \hat{S}_0$ is the zero homomorphism then we can identify $\chi$ with the real number zero. So, assume $\chi : S \mapsto \mathbb{C}$ is a character. Using similar techniques as in the previous question, we can show that $\chi(0) = 1$ and $\chi(n) = [\chi(1)]^n$ for each $n \in \mathbb{N}_0$. Furthermore,

$$\chi(n) = \chi(n^*)$$

$$= \chi(n)^*$$

for each $n \in \mathbb{N}_0$. Therefore, if we let $z = \chi(1)$, then $z = z \in \mathbb{R}$. Conversely, given any $a \in \mathbb{R}$ assigning $\chi(n) = a^n$ for each $n \in \mathbb{N}_0$ yields a homomorphism in $\hat{S}_0$.

1.8 Question. Given $S = (\mathbb{N}_0 \times \mathbb{N}_0, +)$ and $(n, m)^* = (m, n)$, then show that $\hat{S}_0 \cong \mathbb{C}$
Assume \( \chi : S \rightarrow \mathbb{C} \) is a character. Since \( \chi \) is not identically zero, we can show that \( \chi((0,0)) = 1 \). It now follows from the lecture notes on September 20, 2022 that \( \chi((1,0)) = z \in \mathbb{C}/\{0\} \). As in the previous question, we identify the zero homomorphism with zero. Conversely, we can show that if we are given \( z \in \mathbb{C} \) that setting \( \chi((m,n)) = z^m z^n \) for each \( m, n \in \mathbb{N}_0 \) determines a homomorphism in \( \hat{S}_0 \).

We conclude with a spectral theorem for normal operators on finite dimensional Hilbert spaces.

1.9 Theorem. Let \( \mathcal{H} \) be a complex Hilbert space such that \( \dim \mathcal{H} < \infty \). An operator \( A \in B(\mathcal{H}) \) is normal if and only if \( \mathcal{H} = \bigoplus \mathcal{H}_\lambda \) where \( \lambda \) denumerates the eigenvalues of \( A \).

Proof. If \( A \) is normal, then we define an involutive semigroup representation for
\[
S = \{(n, m) : n, m \in \mathbb{N}_0\}
\]
with \((n, m)^* = (m, n)\) by
\[
\pi(n, m) = A^n (A^*)^m.
\]
Since \( S \) is abelian, the representation decomposes into a direct sum of one-dimensional ones on invariant subspaces that are mutually orthogonal. Hence, \( A \) is diagonalizable and the eigenspaces of \( A \) are invariant subspaces.

Conversely, if \( \mathcal{H} \) is a direct sum of eigenspaces for \( A \), taking \( v \in \mathcal{H}_\lambda \), i.e. \( Av = \lambda v \), then by \( A|_{\text{span}(v)} = \lambda \text{id}_{\text{span}(v)} \), we have \( A^*|_{\text{span}(v)} = \overline{\lambda} \text{id}_{\text{span}(v)} \). On each eigenspace, the restriction of \( A \) and \( A^* \) commute, so by direct sum decomposition, \( AA^* = A^*A \). Hence, \( A \) is normal.

2 Banach algebras and Spectral Theory

2.1 Definition. 1. A complex vector space \( A \) with a map \( A \times A \rightarrow A \), \((x, y) \mapsto xy\) is called an (associative) algebra if \((xy)z = x(yz)\) for each \( x, y, z \in A \).

2. Let \( A \) be an algebra. An element 1 is called a unit if \( 1a = a1 = a \) for each \( a \in A \).

3. Let \( A \) be an associative algebra with a unit. An element \( a \in A \) is called invertible if there is \( b \in A \) such that \( ab = ba = 1 \). In that case, the inverse is unique. We then say \( b \) is the inverse of \( a \), \( b = a^{-1} \).

4. An algebra \( A \) which is a Banach space is called a Banach algebra if \( \|ab\| \leq \|a\| \|b\| \) for \( a, b \in A \).

2.2 Remark. The set \( G(A) \) of invertible elements forms a group with unit 1.

2.3 Claim. Let \( A \) be an associative algebra with a unit. If \( a \in A \) is invertible then the inverse is unique.
Proof. Suppose $a$ is invertible and that $b_1$ and $b_2$ are inverses of $A$. Observe that,

\[
    b_1 = b_1 1 \\
    = b_1 (ab_2) \\
    = (b_1 a) b_2 \\
    = 1 b_2 \\
    = b_2
\]

This shows the inverse is unique. $\Box$

2.4 Lemma. Multiplication in a Banach algebra is continuous.

Proof. Let $a_n \to a$ and $b_n \to b$. Using the definition of a Banach algebra and the triangle inequality, we have

\[
    \|a_n b_n - ab\| = \|a_n b_n - ab_n + ab_n - ab\| \\
    \leq \|a_n b_n - ab_n\| + \|ab_n - ab\| \\
    \leq \|a_n - a\| \|b_n\| + \|a\| \|b_n - b\|
\]

Note $(\|b_n\|)_{n=1}^{\infty}$ is bounded due to the assumption that $b_n \to b$. Using this and that the two sequences converge by assumption, it follows that $\|a_n b_n - ab\| \to 0$. $\Box$

2.5 Definition. 1. An involutive algebra $A$ is a (associative) complex algebra for which there is a map $a \mapsto a^*$ such that for each $a, b \in A, \lambda, \mu \in \mathbb{C}$,

(a) $(a^*)^* = a$, 
(b) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$
(c) $(ab)^* = b^* a^*$

2. If $(A, \| \cdot \|)$ is a Banach algebra with involution and $\|a^*\| = \|a\|$ for each $a \in A$, then we say that $A$ is a Banach-*-algebra.

3. If $(A, \| \cdot \|)$ is a Banach-*-algebra with the property that $\|\lambda a^*\| = \|a\|^2$ for each $a \in A$, then this is called a C*-algebra.

4. If $(A, *)$ is an involutive algebra, then $\hat{A}$ is the set of non-zero homomorphism of $A$ to $\mathbb{C}$.

5. If $A$ is a Banach-*-algebra, then we write $\hat{A}$ for the set containing the continuous non-zero homomorphisms.

6. An element $a \in A$, $A$ an involutive algebra, is called

(a) normal if $a a^* = a^* a$,
(b) Hermitian if $a = a^*$,
(c) orthogonal projection if $a a^* = a$.

2.6 Remark. If $\mathcal{H}$ is a complex Hilbert space and $A \subset B(\mathcal{H})$ is a closed subset which forms an algebra with the adjoint as involution, $a \mapsto a^*$, i.e. $A^* \subset A$, then $A$ is a C*-algebra. In particular, $B(\mathcal{H})$ is a C*-algebra. This is the case because on $B(\mathcal{H})$, we had shown $\|a^* a\| = \|a\|^2 = \|a^*\|^2$ for each $a \in B(\mathcal{H})$. 

3