Lecture Notes from October 06, 2022
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Last time

- The spectral theorem for normal operator on finite dimensional Hilbert spaces,
- Involution algebras, Banach-*-algebra, C*-algebra

Warm up:

2.2 Question. If $A$ is involutive algebra, and $1$ is a left unit i.e. $1 \alpha = \alpha$ for each $\alpha \in A$, then show $1$ is unique left unit and it is also right unit.

- **right unit**: we start from $11^* = 1^*$, taking $(.)^*$ both sides, we obtain

$$ (11^*)^* = (1^*)^* \implies 11^* = 1 $$

Thus $1 = 1^*$. We then have $\alpha 1 = (1^* \alpha^*)^* = (\alpha^*)^* = \alpha$.

- **uniqueness**: Now we see if there is (another) left unit $e$, then

$$ e = e 1 \quad e \text{ is right unit} \implies e 1 = 1 \quad e \text{ is left unit} $$

2.3 Lemma. Let $A$ be an involutive algebra, then the following properties hold:

1. Hermitian elements in $A$ are normal,
2. An element of the form $xx^*$ for $x \in A$, is Hermitian
3. The product of two Hermitian elements $x$ and $y$ is Hermitian if and only if $xy = yx$
4. $A = A_n \oplus iA_n$ i.e. each $\alpha \in A$ has a unique decomposition $\alpha = b + ic$ with $b, c$ Hermitian
5. An element $\alpha = b + c$, $b = b^*$, $c = c^*$ is normal if and only if $bc = cb$
6. $A$ has a unit, and $x$ has an inverse, then $(x^{-1})^* = (x^*)^{-1}$
7. If $\| \|$ is a sub-multiplicative norm on $A$ and $\|x\|^2 \leq \|x^* x\|$ for each $x \in A$, then $\|x^*\| = \|x\|$ and $\|x^* x\| = \|x\|^2$

2.4 Remark. $iA_n$ is all the skew Hermitian with $A_n$ is Hermitian
Proof.

(1) Let \( a \in A \) be Hermitian i.e \( a = a^* \), then \( a a^* = a a = a^* a = a^* a \). So \( a \) is normal.

(2) Let \( x \in A \). Then we have \((xx^*)^* = (x^*)^* x^* = xx^*\), so \( xx^* \) is Hermitian.

(3) Let \( x, y \in A \) be Hermitian.
If \( xy = yx \), then \((xy)^* = (yx)^* = x^* y^* = xy \) since \( x = x^* \), \( y = y^* \).
Conversely, if \( xy \) is Hermitian, then \( xy = (xy)^* = y^* x^* = yx \) since \( x = x^* \), \( y = y^* \).

(4) Given \( a \in A \), we write \( b = \frac{a + a^*}{2} \) and \( c = \frac{a - a^*}{2i} \). Then \( b, c \) are Hermitian and \( a = b + ic \).
Moreover, if \( a = b' + ic' \) with \((b')^* = b' \) and \((c')^* = c' \) then by taking Hermitian and antiHermitian parts give \( b' = b \) and \( c' = c \).

(5) We have
\[
\begin{align*}
  a a^* &= (b + ic)(b - ic) \\
        &= b^2 + i c b - i b c + c^2 \\
  a^* a &= (b - ic)(b + ic) \\
        &= b^2 - i c b + i b c + c^2
\end{align*}
\]
By comparing these expressions, \( a^* a = a a^* \) if and only if \( cb - bc = 0 \) or \( cb = bc \).

(6) If \( A \) has a unit \( 1 \) and \( x \) is invertible. Then
\[x^{-1} x = x x^{-1} = 1\]
then applying the involution,
\[x^* (x^{-1})^* = (x^{-1})^* x^* = (x^{-1} x)^* = 1^* = 1\]
Hence \( x^* \) has an inverse which can be identified as \((x^{-1})^*\).

(7) First, we note that for \( x \in A \),
\[
\| x \|^2 \leq \| x^* x \| \leq \| x^* \| \| x \|
\]
So we know, \( \| x \| \leq \| x^* \| \). Thus, implies that
\[
\| x \| \leq \| x^* \| \leq \| (x^*)^* \| = \| x \|
\]
Hence, equality holds throughout. Returning to the first chain of inequality gives
\[
\| x \|^2 \leq \| x^* x \| \leq \| x^* \| \| x \| \leq \| x \| \| x \| = \| x \|^2
\]
So the quality holds between \( \| x \|^2 \) and \( \| x^* x \| \)
\[ \square \]
2.5 Example (for $C^*$-algebra). Let $X$ be a locally Hausdorff space, $C_0(X)$ is the set of the continuous on $X$ such that for each $e > 0$, there is a compact set $K$, if $x \notin K$, $|f(x)| < e$. We equip $C_0(X)$ with a norm

$$
\|f\|_\infty = \sup_{x \in X} |f(x)|
$$

This is a closed subspace of the bounded, continuous functions on $X$. With $f^*(x) = \overline{f(x)}$, this becomes $C^*$-algebra.

Proof:

- Completeness:
  Consider a Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset C_0(X)$. Then we have

$$
\|f_n - f_m\|_\infty = \sup_{x \in X} |f_n(x) - f_m(x)| \to 0 \text{ as } n, m \to \infty
$$

Define $f : X \to \mathbb{C}$ as $f(x) = \lim_{n \to \infty} f_n(x)$. Then $|f_n(x) - f(x)| \to 0$ as $n \to \infty$

First, we show that $\lim_{n \to \infty} f_n = f$. We have that

$$
\|f - f_n\|_\infty = \sup_{x \in X} |f(x) - f_n(x)| = \|f - f_n\|_\infty = \sup_{x \in X} |f(x) - f_m(x) + f_m(x) - f_n(x)|
$$

$$
\leq \sup_{x \in X} |f(x) - f_m(x)| + \sup_{x \in X} |f_m(x) - f_n(x)|
$$

$$
\to 0 \quad \text{as } n, m \to \infty
$$

Next, we show that $f \in C_0(X)$.

Since $f_n \in C_0(X)$, then $f_n$ is continuous at $x \in X$. Given $e > 0$, there is $\delta > 0$ such that for all $y \in X$, $\|x - y\| < \delta$, implies that $|f_n(x) - f_n(y)| < e$. With $x \in X$ and same condition such that $\|x - y\| < \delta$, we have

$$
|f(x) - f(y)| = |\lim_{n \to \infty} f_n(x) - \lim_{n \to \infty} f_n(y)|
$$

$$
\leq \lim_{n \to \infty} |f_n(x) - f_n(y)| < e
$$

Thus, $f$ is continuous on $X$. Hence $f \in C_0(X)$ because for each $e > 0$, there is a compact set $K$, such that if $x \notin K$, $|f(x)| = |\lim_{n \to \infty} f_n(x)| \leq \lim_{n \to \infty} |f_n(x)| < e$ (same compact set $K$ in $f_n \in C_0(X)$ condition).

- Now, we show that $C_0(X)$ is an algebra. Let $f, g \in C_0(X)$, then

$$
\|f \cdot g\|_\infty = \sup_{x \in X} |f(x)g(x)| \leq \sup_{x \in X} \left(\sup_{x \in X} |f(x)||g(x)|\right)
$$

$$
= \sup_{x \in X} |f(x)| \left(\sup_{x \in X} |g(x)|\right) = \|f\|_\infty \|g\|_\infty
$$
• Next, for \( f \in C_0(X) \), \( f^*(x) = \overline{f(x)} \). We have
\[
\|f^*\|_\infty = \sup_{x \in X} |f^*(x)| = \sup_{x \in X} |\overline{f(x)}| = \sup_{x \in X} |f(x)| = \|f\|_\infty
\]

• Finally, we show \( C_0(X) \) is \( C^* \)-algebra. For each \( f \in C_0(X) \), consider
\[
f^* \cdot f(x) = f^*(x) f(x) = \overline{f(x)} f(x) = |f(x)|^2
\]

Taking sup over \( X \), we get
\[
\|f^* \cdot f\|_\infty = \sup_{x \in X} |f(x)|^2 \geq \left( \sup_{x \in X} |f(x)| \right)^2 = \|f\|_\infty^2
\]

Hence by Lemma 2.3(7), this completes the proof. \( \square \)

For this \( C^* \)-algebra, the map \( \delta_x : C_0(X) \to \mathbb{C} \) with \( f \mapsto f(x) \) is a (nontrivial) character on \( C_0(X) \). This is because of Urysohn’s lemma which guarantees the existence of a function \( f \in C_0(X) \) with \( f(x) = 1 \). We will see later, \( (C_0(X)) = \{ \delta_x : x \in X \} \).

As a special example, if \( X = \mathbb{N} \), \( C_0(X) = \mathbb{C} \) and \( \mathcal{C}_0 = \{ \delta_n : n \in \mathbb{N} \} \equiv \mathbb{N} \).

More examples with different types of norm.

2.6 Examples. Let \( S \) be an involutive semigroup. Consider \( \ell^1(S) \) i.e. the space of all \( f : S \to \mathbb{C} \) with \( \|f\|_1 = \sum_{s \in S} |f(s)| < \infty \). (Note that the set \( \{ s \in S : f(s) \neq 0 \} \) is at most countable).

Equip \( \ell^1(S) \) with the convolution
\[
(f \ast g)(s) = \sum_{a,b \in S \atop ab = s} f(a)g(b)
\]
and let \( f^*(s) = \overline{f(s^*)} \). Then \( \ell^1(S) \) becomes a Banach-*-algebra.

Proof.

• First, we see that \( \ell^1(S) \) is closed under convolution. Let \( f, g \in \ell^1(S) \). Then
\[
\|f\|_1 = \sum_{s \in S} |f(s)| < \infty \quad \text{and} \quad \|g\|_1 = \sum_{s \in S} |g(s)| < \infty
\]

Let \( J_f = \{ s \in S : f(s) \neq 0 \} \) and \( J_g = \{ s \in S : g(s) \neq 0 \} \). Note that \( J_f \) and \( J_g \) are at most countable. Consider
\[
\|f \ast g\|_1 = \sum_{s \in S} |f \ast g(s)| = \sum_{s \in S} \left| \sum_{a,b \in S \atop ab = s} f(a)g(b) \right|
\]
\[
\leq \sum_{s \in S} \left( \sum_{a,b \in S \atop ab = s} |f(a)| |g(b)| \right) = \sum_{a \in S} \left( |f(a)| \sum_{b \in S \atop ab = s} |g(b)| \right) \quad (1)
\]
\[
\leq \sum_{a \in S} \left( |f(a)| \|g\|_1 \right) = \|g\|_1 \|f\|_1 < \infty
\]

Thus, \( f \ast g \in \ell^1(S) \).
• Next, we show that \( \ell^1(S) \) is a Banach algebra. Consider a Cauchy sequence \( \{f_n\}_{n \in \mathbb{N}} \subset \ell^1(S) \). Define \( f : S \to \mathbb{C} \), \( f(s) = \lim_{n \to \infty} f_n(s) \).

\[
\|f - f_n\|_1 = \sum_{s \in S} |f(s) - f_n(s)| = \sum_{s \in S} |f(s) - f_m(s) + f_m(s) - f_n(s)|
\]

\[
\leq \sum_{s \in S} |f(s) - f_m(s)| + \sum_{s \in S} |f_m(s) - f_n(s)|
\]

since \( f(s) = \lim_{m \to \infty} f_m(s) \) and \( f_n \) is Cauchy

\[
\longrightarrow 0 \quad \text{as } n, m \to \infty
\]

Also,

\[
\|f\|_1 = \sum_{s \in S} |f(s)| = \sum_{s \in S} \lim_{n \to \infty} |f_n(s)| \leq \lim_{n \to \infty} \sum_{s \in S} |f_n(s)| < \infty
\]

Thus, \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \in \ell^1(S) \). So \( \ell^1(S) \) is a Banach space and we then even have \( \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1 \) as in equation 1. Hence, it is a Banach algebra.

• And finally, we show \( \ell^1(S) \) is a Banach-*-algebra. Let \( f \in \ell^1(S) \) and \( f^*(s) = \overline{f(s^*)} \). Then

\[
\|f^*\|_1 = \sum_{s \in S} |f^*(s)| = \sum_{s \in S} |\overline{f(s^*)}| = \sum_{s \in S} |f(s^*)|
\]

\[
= \sum_{s^* \in S} |f(s)| = \|f\|_1
\]

We then have a homomorphism \( \eta : S \to \ell^1(S) \) that maps \( s \mapsto \delta_s \) with

\[
\delta_s(t) = \begin{cases} 
1 & \text{if } s = t \\
0 & \text{elsewhere}
\end{cases}
\]

For these,

\[
(\delta_s \ast \delta_t)(x) = \sum_{a,b \in S} \delta_s(a) \delta_t(b)
\]

\[
= \begin{cases} 
1 & \text{if } \delta_s(a) = 1 = \delta_t(b) \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
1 & \text{if } s = a, t = b \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
1 & \text{if } ab = st = x \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \delta_{st}(x)
\]

Since

\[
\delta_s^*(t) = \overline{\delta_s(t^*)} = \delta_s(t^*) = \begin{cases} 
1 & \text{if } s = t^* \text{ and } t^* = \overline{s^*} \text{ is the involution of } S \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
1 & \text{if } s^* = (t^*)^* = t \\
0 & \text{otherwise}
\end{cases}
\]

then \( \delta_s^* = \delta_{s^*} \), we even have \( \eta(s^*) = \delta_{s^*} = \delta_s = (\eta(s))^* \), so \( \eta \) is a homomorphism that identifies the involutive semigroup with a subset of \( \ell^1(S) \), so it embeds \( S \) in the Banach-*-algebra.