Lecture Notes from October 13, 2022
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Last Time
Banach $\ast$ algebras and characters by examples.

Warm up: Consider $l^1(\mathbb{Z})$ as Banach $\ast$ algebra with $x \ast y$ defined by convolution and $(x^*)_k = (x_{-k})$ for $x \in l^1(\mathbb{Z}), k \in \mathbb{Z}$.

4.5 Question. What are the characters on this Banach $\ast$ algebra?

Again, we use $\eta : k \to \delta_k$, then for $f \in l^1(\mathbb{Z})$, a (bounded) character $\chi$,

$$
\chi(f) = \chi\left(\sum_{k \in \mathbb{Z}} f(k)\eta(k)\right) = \sum_{k \in \mathbb{Z}} f(k)\chi(\eta(k))
$$

and by $\|\chi\| < \infty$, $(l^1(\mathbb{Z}))' = l^\infty(\mathbb{Z})$, we know that $\chi \circ \eta$ is bounded and $\chi \circ \eta$ is a character on $\mathbb{Z}$ from the embedding of $\mathbb{Z}$ in the algebra. Hence, determined by $\chi \circ \eta(1) = z \in S^1 = \{w : |w| = 1\}$. Conversely, if $\gamma$ is a character on $\mathbb{Z}$, then $\chi(f) = \sum_{k \in \mathbb{Z}} f(k)\gamma(k)$ defines a character on $l^1(\mathbb{Z})$.

We summarize, $l^1(\mathbb{Z}) \cong \hat{\mathbb{Z}} \cong S^1$.

Consequently, we can map $l^1(\mathbb{Z})$ to a space in $C(S^1)$, using that each character $\gamma$ of $\mathbb{Z}$ is of the form $k \mapsto \mathbb{Z}^k$, hence we can define $\hat{f}(z) = \sum_{k \in \mathbb{Z}} f(k)z^k$.

Next, we will see that we will see that we can relate Banach $\ast$ Algebras with the Fourier transform in a similar way.

4.6 Example. We first construct a algebra with involution. Let $C_c(\mathbb{R}^n)$ be the space of continuous functions with compact support. Define $f^*(x) = \overline{f(-x)}$ and for $x \in \mathbb{R}^n; f, g \in C_c(\mathbb{R}^n)$; $(f \ast g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)d\lambda(y)$, where $\lambda$ is a Lebesgue measure. For such $f$ and $g$, $f \ast g$ is again in $C_c(\mathbb{R}^n)$. This is because, since $g$ is continuous so for given $\epsilon > 0$, $\exists \delta > 0$ such that $||x - y|| < \delta \implies |g(x) - g(y)| < \epsilon$.

Now, $f$ is continuous on a compact set, this implies $f$ is bounded. Therefore, $\exists M > 0$ such that
\[\|f(x)\| < M. \text{ Consider,}\]

\[
\|f \ast g(x) - f \ast g(y)\| = \left\| \int_{\mathbb{R}^n} f(z)[g(x - z) - g(y - z)]d\lambda(z) \right\|
\leq \int_{\mathbb{R}^n} \|f(z)\|\|g(x - z) - g(y - z)\|d\lambda(z)
\leq Me \int_{\mathbb{R}^n} d\lambda(z)
= Me\lambda(\mathbb{R}^n)
\]

since \( g \) being continuous and compactly supported is uniformly continuous. Therefore, \( f \ast g \) is continuous. Also, \( \text{Supp}(f \ast g) \subset \text{Supp}(f^\ast) \cup \text{Supp}(g^\ast) \), and since union of two compact sets is compact, \( f \ast g \) is compact. Hence, \( f \ast g \in C_c(\mathbb{R}^n) \). This space forms a commutative involutive algebra.

We show associativity,

\[
((f \ast g) \ast h)(x) = \int_{\mathbb{R}^n} f(x \ast y)(y)h(x - y)d\lambda(y)
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(z)g(y - z)d\lambda(z)h(x - y)d\lambda(y)
\]

\[
\text{Fubini} \Rightarrow \int_{\mathbb{R}^n} f(z) \left( \int_{\mathbb{R}^n} g(y - z)h(x - y)d\lambda(y) \right)d\lambda(z)
\]

\[
y - z = u \Rightarrow \int_{\mathbb{R}^n} f(z) \left( \int_{\mathbb{R}^n} g(u)h(x - u - z)d\lambda(u) \right)d\lambda(z)
\]

\[
= \int_{\mathbb{R}^n} f(z)g \ast h(x - z)d\lambda(z)
= (f \ast (g \ast h))(x)
\]

Moreover,

\[
(f \ast g) \ast (x) = \int_{\mathbb{R}^n} f(y)g(x - y)d\lambda(y)
= \int_{\mathbb{R}^n} \overline{f(y)}g(-x - \overline{y})d\lambda(y)
\]

\[
\overline{u} = -y \Rightarrow \int_{\mathbb{R}^n} \overline{f(u)}g(-x + \overline{u})d\lambda(u)
\]

\[
= \int_{\mathbb{R}^n} f^\ast(u)g^\ast(x - u)d\lambda(u)
= (f^\ast \ast g^\ast)(x)
\]
Together with,
\[(f \ast g)(x) = \int_{\mathbb{R}^n} f(y)g(y - x)d\lambda(y)\]
\[= \int_{\mathbb{R}^n} f(-u)g(x + u)d\lambda(u)\]
\[= \int_{\mathbb{R}^n} g(w)f(x - w)d\lambda(w)\]
\[= (g \ast f)(x)\]

Hence, \(C_c(\mathbb{R}^n)\) forms an involutive algebra.

Next, we define, \(\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|d\lambda(x)\) We claim this norm is sub-multiplicative. To see this, consider \(f, g \in C_c(\mathbb{R}^n)\)
\[\|f \ast g\|_1 = \int_{\mathbb{R}^n} |f(y)g(x - y)d\lambda(y)|d\lambda(x)\]
\[\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)||g(x - y)|d\lambda(y)d\lambda(x)\]
\[\overset{\text{Fubni}}{=} \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |g(x - y)|d\lambda(x)d\lambda(y)\]
\[= \|g\|_1 \int_{\mathbb{R}^n} |f(y)|d\lambda(y)\]
\[= \|g\|_1 \|f\|_1\]

We also observe, \(\|f^*\|_1 = \|f\|_1\). Since
\[\|f^*\|_1 = \int_{\mathbb{R}^n} |f(-x)|d\lambda(x)\]
\[= \int_{\mathbb{R}^n} |f(-x)|d\lambda(x)\]
\[\overset{\text{Fubni}}{=} \int_{\mathbb{R}^n} |f(u)|d\lambda(u)\]
\[= \|f\|_1\]

Now, taking \(L^1(\mathbb{R}^n)\) to be the completion of \(C_c(\mathbb{R}^n)\), then by continuity of \(L^1(\mathbb{R}^n)\), we can extend \(f \ast g\) and \(f \to f^*\) to \(L^1(\mathbb{R}^n)\) since if \(f_n\) and \(g_n\) are two Cauchy sequences in \(C_c(\mathbb{R}^n)\) such that \(f_n \to f\) and \(g_n \to g\) in \(L^1(\mathbb{R}^n)\) and \(g_n\) is continuous on \(B(0, r)\) for some \(r > 0\), we have
\[\|f_n \ast g_n - f_m \ast g_m\|_1 = \int_{\mathbb{R}^n} |f_n(y)g_n(x - y)d\lambda(y) - f_m(y)g_m(x - y)d\lambda(y)||d\lambda(y)\]
\[\leq \int_{\mathbb{R}^n} ||f_n(y) - f_m(y)||g_n(x - y) - g_m(x - y)||_1 d\lambda(y)\]
This implies \(|f_n * g_n - f_m * g_m|_1| \rightarrow 0\) as \(n \rightarrow \infty\) as \(f_n\) and \(g_n\) are Cauchy. Hence, \(f_n * g_n\) is Cauchy and convergent in \(L^1(\mathbb{R}^n)\). Also, \(f_n * g_n\) is uniformly continuous on \(B(0, r)\) since it is continuous and compactly supported. Restricting the convolution to \(B(0, r)\), we get

\[
\|f_n * g_n - f * g\|_1 \leq \int_{B(0, r)} \|f_n(y) - f(y)\| \|g_n(x-y) - g(x-y)\|_1 d\lambda(y)
\]

Since, \(f_n \rightarrow f\) and \(g_n \rightarrow g\), we have \(f_n * g_n \rightarrow f * g\) on \(B(0, r)\). It converges on \(L^1(\mathbb{R}^n)\), by taking the union of all the balls of radius \(r > 0\) and we obtain a Banach \(-\ast-\) algebra. This algebra is also called the \(L^1\)- algebra of \(\mathbb{R}^n\).

Next, we want to study the characters of this algebra. We consider an example

4.7 Example.

\[
\chi_x : \mathbb{R}^n \rightarrow S^1
\]

\[
y \mapsto e^{ixy}
\]

Then, \(\chi_x\) is a continuous non trivial group homomorphism from \(\mathbb{R}^n\) to \(S^1\), hence a character on \(\mathbb{R}^n\). By boundedness of \(\chi_x\), we obtain \(\chi_x = \int_{\mathbb{R}^n} f(y)e^{ix.y}d\lambda(y)\) and we claim \(\chi_x\) defines a character on \(L^1(\mathbb{R})\). Indeed,

\[
\chi_x(f^*) = \int_{\mathbb{R}^n} \overline{f(-y)}e^{ix.y}d\lambda(y)
\]

\[
= \int_{\mathbb{R}^n} f(u)e^{-ix.u}d\lambda(u)
\]

\[
= \int_{\mathbb{R}^n} f(u)e^{ix.u}d\lambda(u)
\]

\[
= \overline{\chi_x(f)}
\]